Persistence of non-compact Normally Hyperbolic Invariant Manifolds

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Introduction

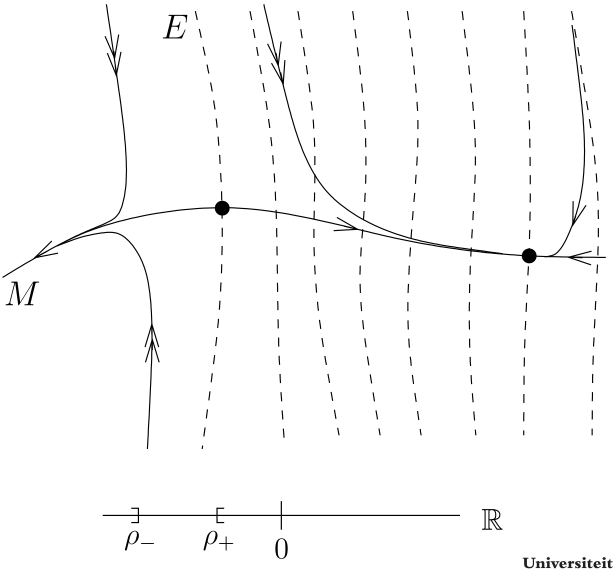
Normally hyperbolic invariant manifolds are generalizations of hyperbolic fixed points. The (invariant) fixed point is replaced by a manifold. Properties shared:

- persistence under small perturbations
- existence of (un)stable manifolds/fibrations.

Major application: reduction of dynamics to the invariant manifold. For example in

- dissipative/diffusive systems
- singular perturbation problems

A normally hyperbolic system



A normally hyperbolic system (cont'd)

Let Q be some ambient space (think: $\mathbb{R}^n$ or Banach space) and $v \in \mathfrak{X}^k(Q)$, $k \geqslant 1$ with flow Φ^t. A submanifold $M \subset Q$ is a NHIM if:

$$\forall\, t \in \mathbb{R} \colon \Phi^t(M) = M$$

$$\mathrm{T}_M Q = \mathrm{T}M \oplus C$$

$$\mathrm{D}\Phi^t_+ \colon \mathrm{T}M \to \mathrm{T}M, \quad \mathrm{D}\Phi^t_- \colon C \to C$$

$$\begin{aligned} \forall\, t \geqslant 0 \colon \left\| \mathrm{D}\Phi^t_- \right\| &\leqslant C_- \, e^{\rho_- \, t} \\ \forall\, t \leqslant 0 \colon \left\| \mathrm{D}\Phi^t_+ \right\| &\leqslant C_+ \, e^{\rho_+ \, t} \end{aligned}$$

and $\rho_- < 0$, $\rho_- < \rho_+$.

Classical results

Classical results [HPS '70, Fenichel '71] using the graph transform:

When M is compact and $\rho_- < k\,\rho_+$ then

decoupling: there exists a unique flow-invariant fibration E over M and $E \in C^k$;

persistence: any small perturbation $v \rightsquigarrow \tilde{v}$ yields a new invariant manifold $\tilde{M} \in C^k$, close to M.

Extensions exist:

- semi-flows in Banach spaces
- additional $C^{k,\alpha}$ Hölder continuity
- non-autonomous systems: integral manifold

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Our persistence result

Based on [Henry '81, Vanderbauwhede '89].

When M is a (non-compact, infinite-dimensional, non-autonomous) NHIM with trivial normal bundle and $\|\tilde{v} - v\|$ is uniformly C^1 small and $C^{k,\alpha}$ bounded, then there exists a unique $\tilde{M}$, which is uniformly $C^{k,\alpha}$ close to M.

- compactness is replaced by uniform estimates
- optimal $C^{k,\alpha}$ smoothness result exhausting the spectral gap condition $\rho_- < (k+\alpha)\,\rho_+$
- generalization to a non-trivial normal bundle is WiP

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Construction of the Perron-like contraction

The standard Perron integral operator only works for linearization in a neighborhood around a fixed point. Here, M is global and solutions cannot be localized.

Instead we consider $M \hookrightarrow M \times Y$ and curves $x(t), y(t)$. Now only linearize the Y component:

$$\tilde{v}(x, y) = \tilde{v}_x(x, y) + D_y \tilde{v}(x, 0) \cdot y + r(x, y),$$

$$\begin{aligned} T_x \colon y(t), x_0 &\mapsto x'(t) = \Phi_M(t, t_0, x_0, y), \\ T_y \colon x(t), y(t) &\mapsto y'(t) = \int_{-\infty}^{t} \Psi_Y(t, \tau, x) \cdot r(\tau, x(\tau), y(\tau))\, \mathrm{d}\tau. \end{aligned}$$

Now $T = T_y \circ T_x$ is a contraction leading to $\tilde{M}$.

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