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Existence, stability and interaction of some nonlinear waves

EURANDOM, NDNS+ workshop, April 2010

Jens Rademacher (CWI Amsterdam)

Deviation from Abstract

I will not talk about stability boundaries of wave trains,

see Sjors van der Stelt’s talk Thursday.

Topics

1. Riemann solvers and undercompressive shocks of convex FPU lattices
2. Bifurcations from heteroclinic networks with periodic orbits and tangencies
3. Transition to chaos in the wake of predator invasion
4. First and second order semi-strong interaction

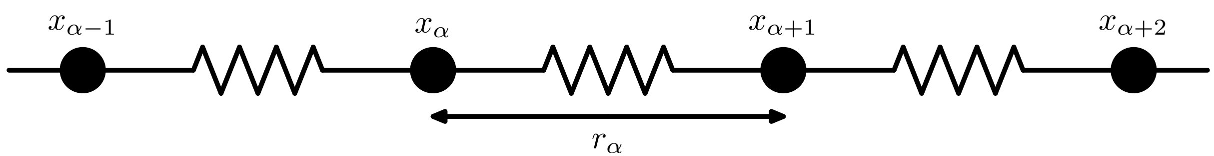
Topic 1

Riemann solvers and undercompressive shocks of convex FPU chains

joint work with M. Herrmann (Oxford)

FPU-type chain

Caricature model for solids or crystals.



Atomic position $\quad x_\alpha(t)$

Particle index $\quad \alpha = 1, \ldots, N$ or $\alpha \in \mathbb{Z}$

Convex potential $\Phi \quad \Rightarrow \quad$ monotone force Φ'

Newton equations for nearest neighbour interaction

$$\ddot{x}_\alpha = \Phi'(x_{\alpha+1} - x_\alpha) - \Phi'(x_\alpha - x_{\alpha-1})$$

Distances $\quad r_\alpha = x_{\alpha+1} - x_\alpha \quad \Rightarrow \quad \dot{r}_\alpha = v_{\alpha+1} - v_\alpha$.

Velocities $\quad v_\alpha = \dot{x}_\alpha \quad \Rightarrow \quad \dot{v}_\alpha = \Phi'(r_\alpha) - \Phi'(r_{\alpha-1})$.

Energy equation

$$\partial_{\bar{t}} \left(\frac{1}{2} v_\alpha^2 + \Phi(r) \right) = v_{\alpha+1} \Phi'(r_\alpha) - v_\alpha \Phi'(r_{\alpha-1}).$$

Macroscopic thermodynamic limit?

Consider $\varepsilon = 1/N \to 0$ with

hyperbolic space-time scaling $\bar{\alpha} = \varepsilon\alpha \quad , \quad \bar{t} = \varepsilon t$

Does not scale amplitude!

Naively: Assume limiting fields $r(\bar{t}, \bar{\alpha})$, $v(\bar{t}, \bar{\alpha})$

$$\begin{aligned} \varepsilon\dot{r}(\bar{t}, \bar{\alpha}) = v(\bar{t}, \bar{\alpha} + \varepsilon) - v(\bar{t}, \bar{\alpha}) \quad &\Rightarrow \quad \partial_{\bar{t}} r \quad = \quad \partial_{\bar{\alpha}} v \\ \ldots \quad &\Rightarrow \quad \partial_{\bar{t}} v \quad = \quad \partial_{\bar{\alpha}}(\Phi'(r)) \end{aligned}$$

'p-system': mass and momentum conservation.

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Conversely: FPU is dispersive discretisation of p-system.

FPU and p-system

How much about FPU can we understand through the p-system?

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Does FPU also behave this way for small ε?

$\Rightarrow$ Solve discrete Riemann problems and identify building blocks.

In particular shocks: for p-system additional selection criterion required.

Shocks (= jump discontinuities)

Hyperbolic theory for p-system emposes ‘kinetic relation’ to select shocks from family of jump-discontinuities that generate a weak solution (Rankine-Hugeniot conditions).

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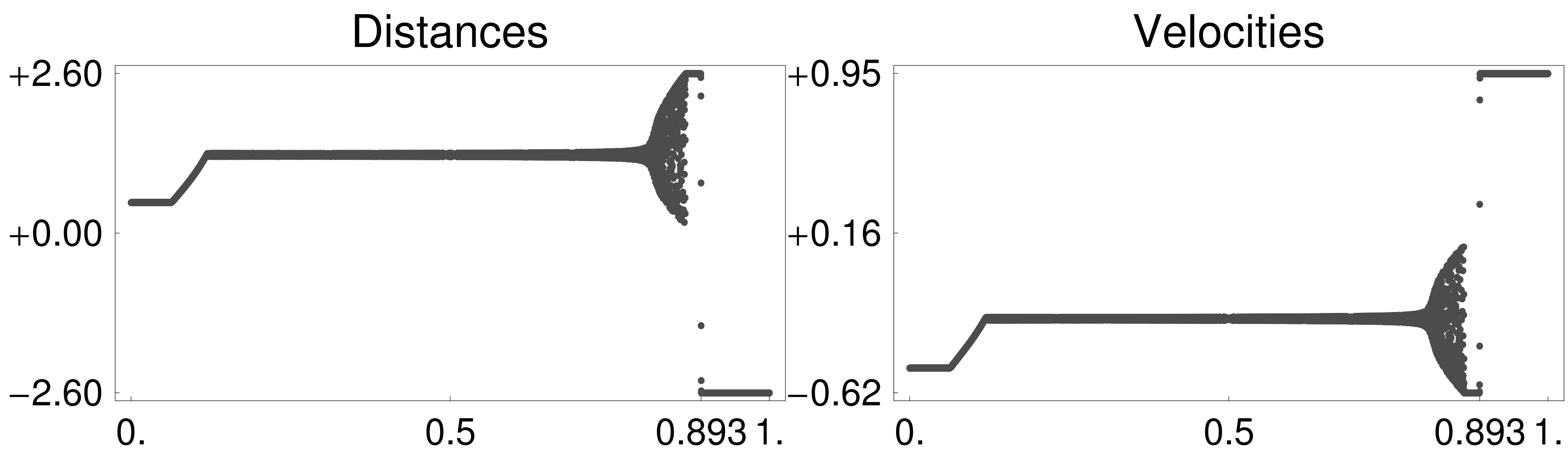
Microscopic energy conservation $\Rightarrow$ cannot expect Lax-shocks.

A typical Riemann problem



Approximately self-similar: all depends essentially on $c = x/t$.

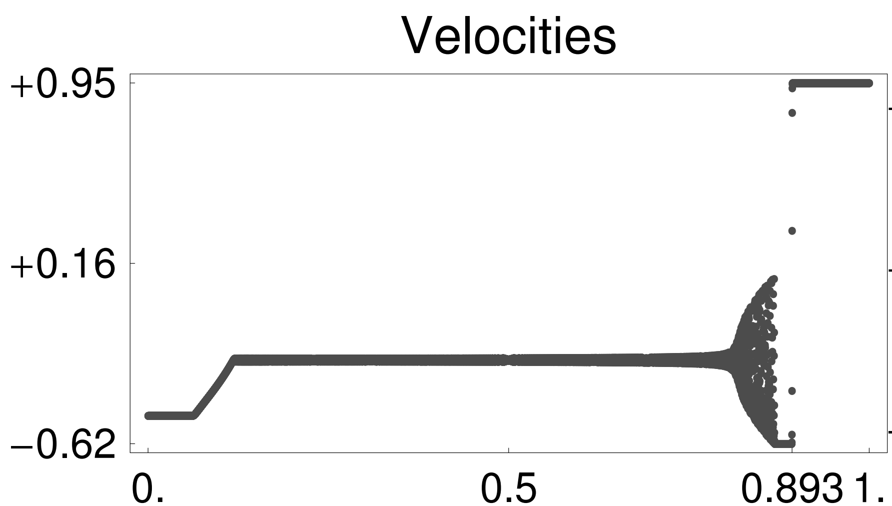
Observe three types of building blocks ('elementary waves'):

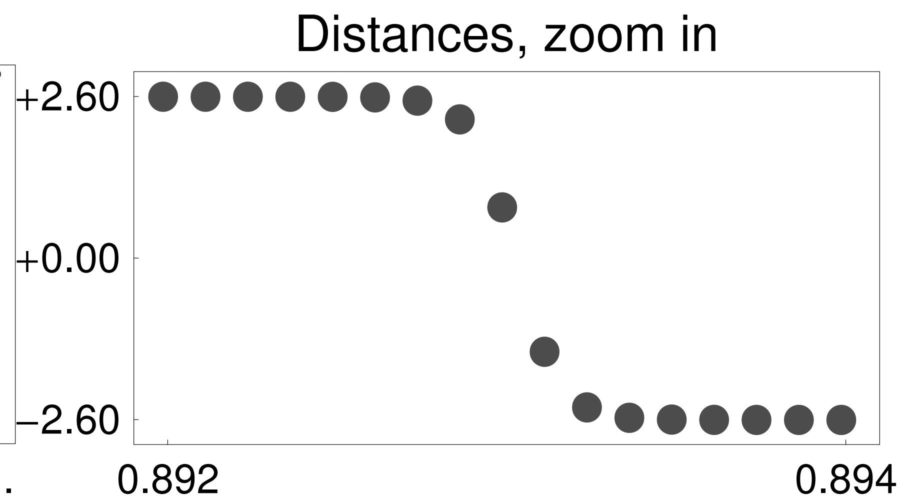
- Rarefaction fan (leftmost)

- Highly oscillatory region (middle) - 'dispersive shock'

- Jump discontinuity (rightmost) - 'conservative shock'

Conservative shock





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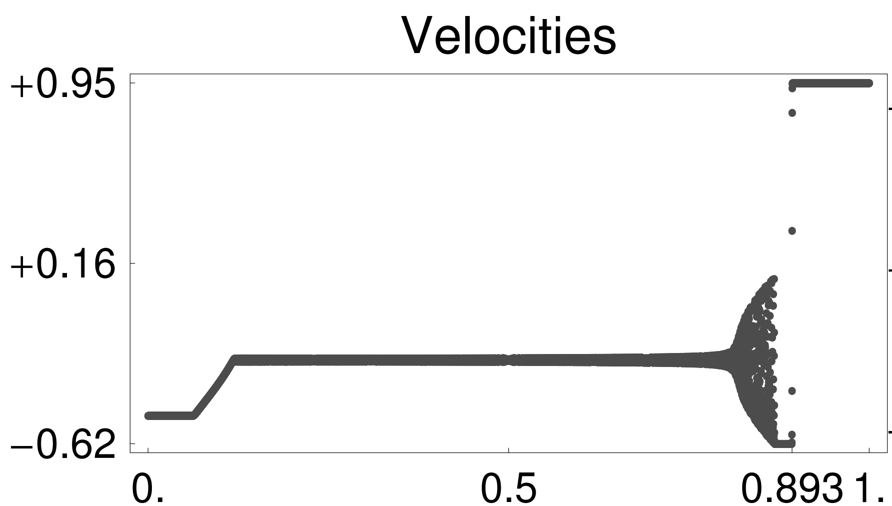
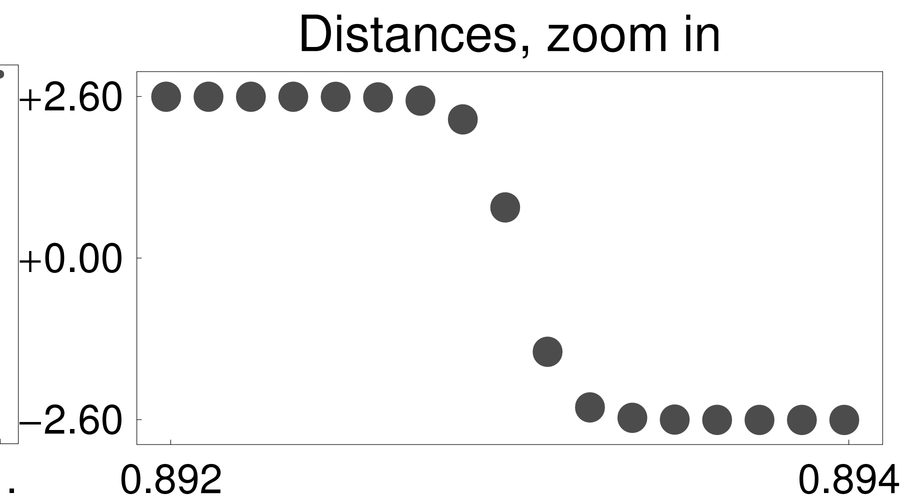
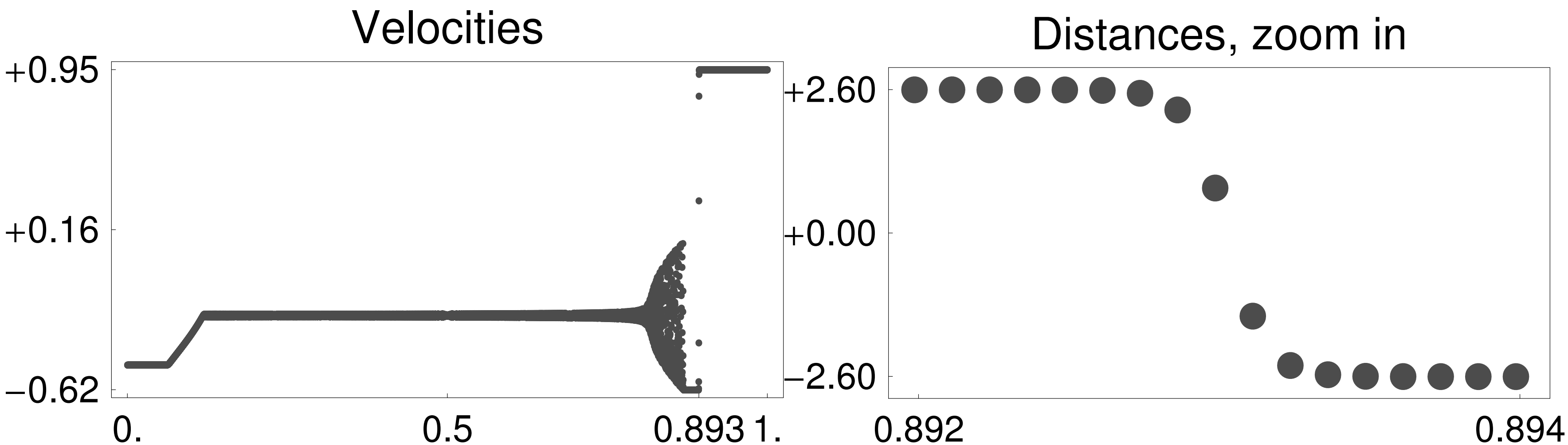




Heteroclinic travelling wave of infinite FPU (Iooss '00) ?

Conservative shock

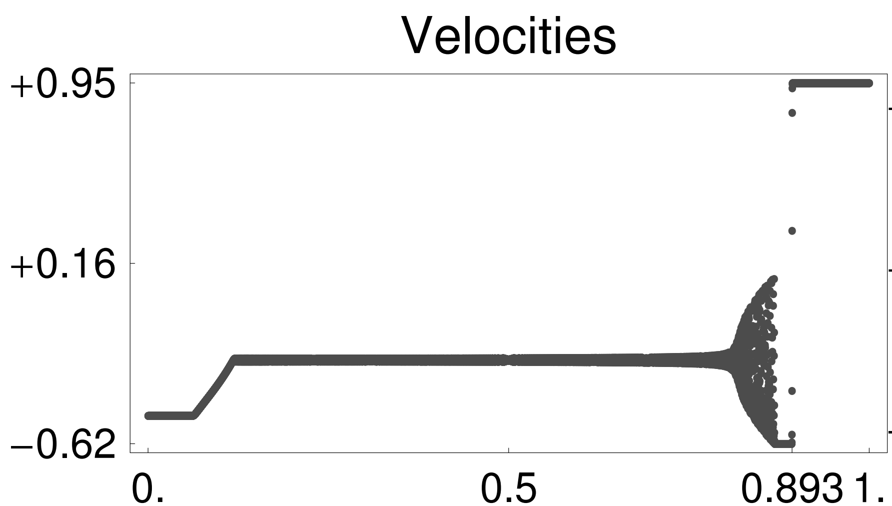


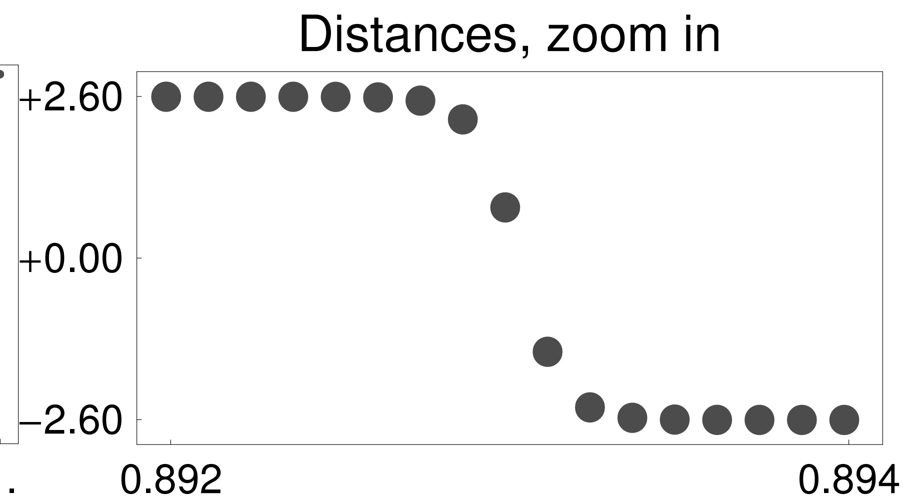
Heteroclinic travelling wave of infinite FPU (Iooss ’00) ?

Theorem [Herrmann, R.]

1. Fronts satisfy jump relations of p-system including energy conservation: ‘conservative shocks’.

Conservative shock





Heteroclinic travelling wave of infinite FPU (Iooss '00) ?

Theorem [Herrmann, R.]

1. Fronts satisfy jump relations of p-system including energy conservation: 'conservative shocks'.

2. Supersonic conservative p-system shocks that satisfy an area condition correspond to fronts. This requires a turning point of the flux.

Proof: Novel variational approach for deviation from p-system shock..

Riemann solver ?

Can we characterise a Riemann solver for FPU that predicts which types of waves and which intermediate states will be taken for given Riemann problem data?

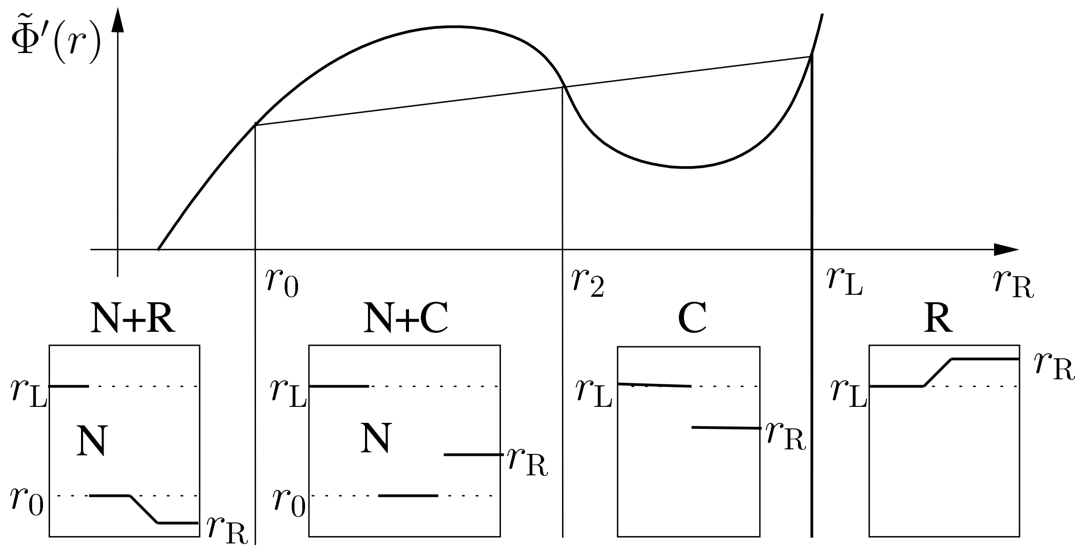
Idea: Adapt/extend p-system Riemann solvers.

Problems:

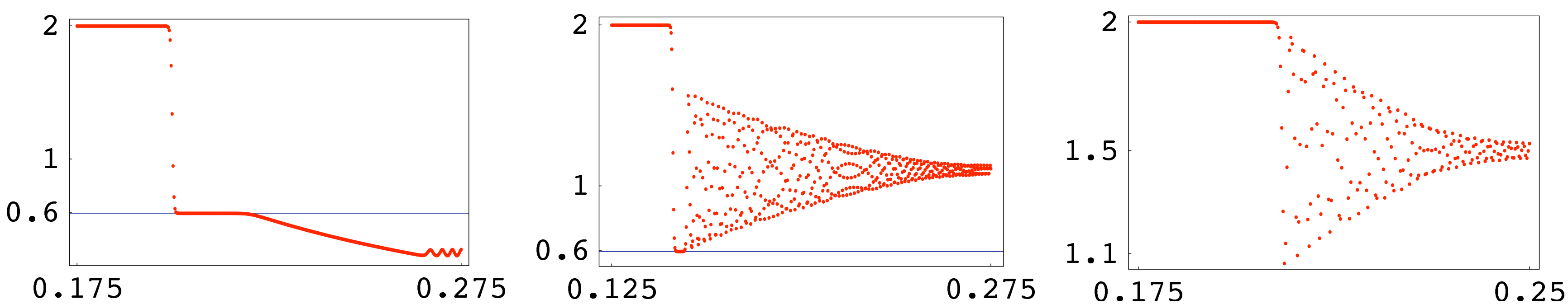
1. Conservative shocks require turning point of flux $\rightarrow$ cannot use classical Lax-theory. Instead 'kinetic relation' theory of LeFloch, Truskinovsky, ...

2. Only supersonic conservative shocks appear in FPU, not subsonic ones - need different solvers!

Shock curve



$\tilde{\Phi}'(r)$
r_0
r_2
r_L
r_R
N+R
N+C
C
R
N
N
r_L
r_0
r_R



2
1
0.6
0.175
0.275
2
1
0.6
0.125
0.275
2
1.5
1.1
0.175
0.25

Outlook

Stability of fronts?

Bifurcation at transition to non-supersonic?

Rigorous Riemann solver for ‘dispersive p-system’?

References (with M. Herrmann):

- Riemann solvers: Nonlinearity 23 (2010) 277-304.

- Existence of fronts: to appear in SIAM J. Math. An.

Topic 2

Bifurcations from heteroclinic networks with periodic orbits and tangencies

partly joint work with A. Champneys (Bristol), V. Kirk (Auckland),

E. Knobloch (Berkeley), B. Oldeman (Montreal)

ODE Setup

$$\frac{\mathrm{d}}{\mathrm{d}x}u(x) = f(u(x);\mu), \quad u(x) \in \mathbb{R}^n,$$

parameters $\mu \in \mathbb{R}^d$, sufficiently large d.

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Heteroclinic network: directed graph, vertices p_i^* equilibria or periodic orbits, edges q_i^* heteroclinic connections.

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Starting point: an arbitrary path, possibly with repetitions

$(p_j)_{j\in J}$, $(q_j)_{j\in J^\circ := J\setminus \min J}$, $\quad q_j$ heteroclinic from p_{j-1} to p_j.

(homoclinic if $p_{j-1} = p_j$) and J either $\mathbb{N}$, $\mathbb{Z}$ or finite.

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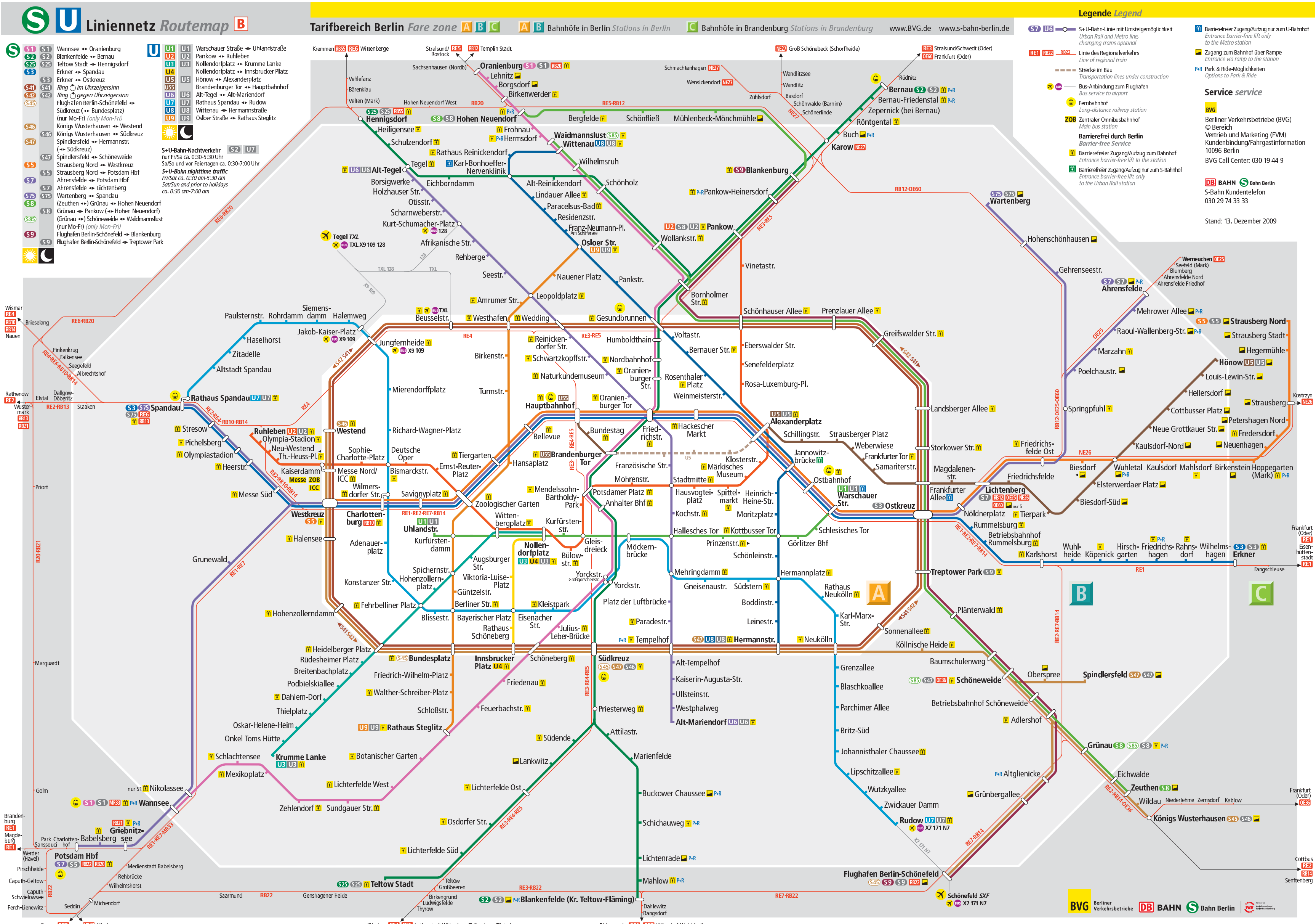
Novelty: allow for periodic orbits in general and tangencies.

Practical example!



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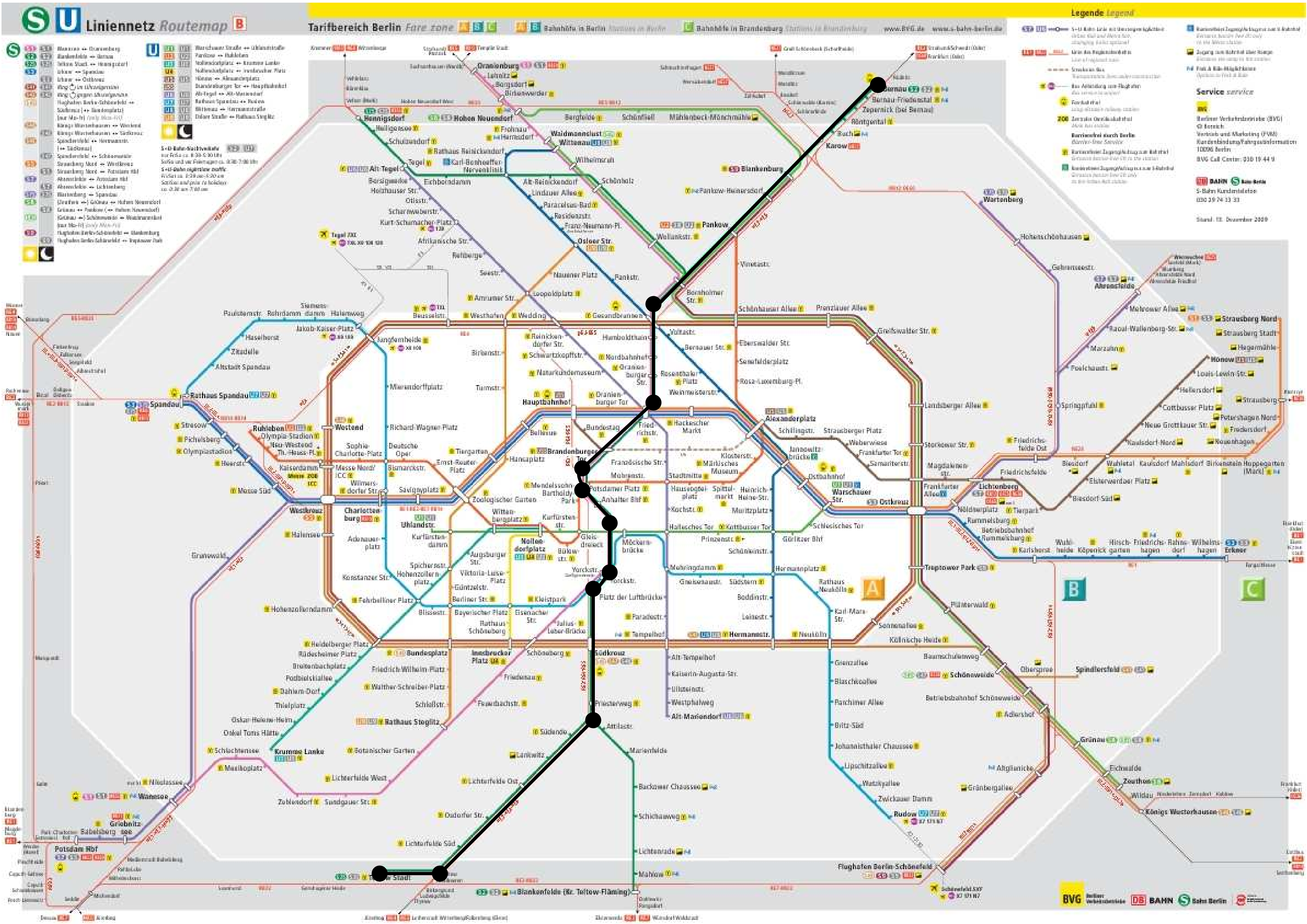

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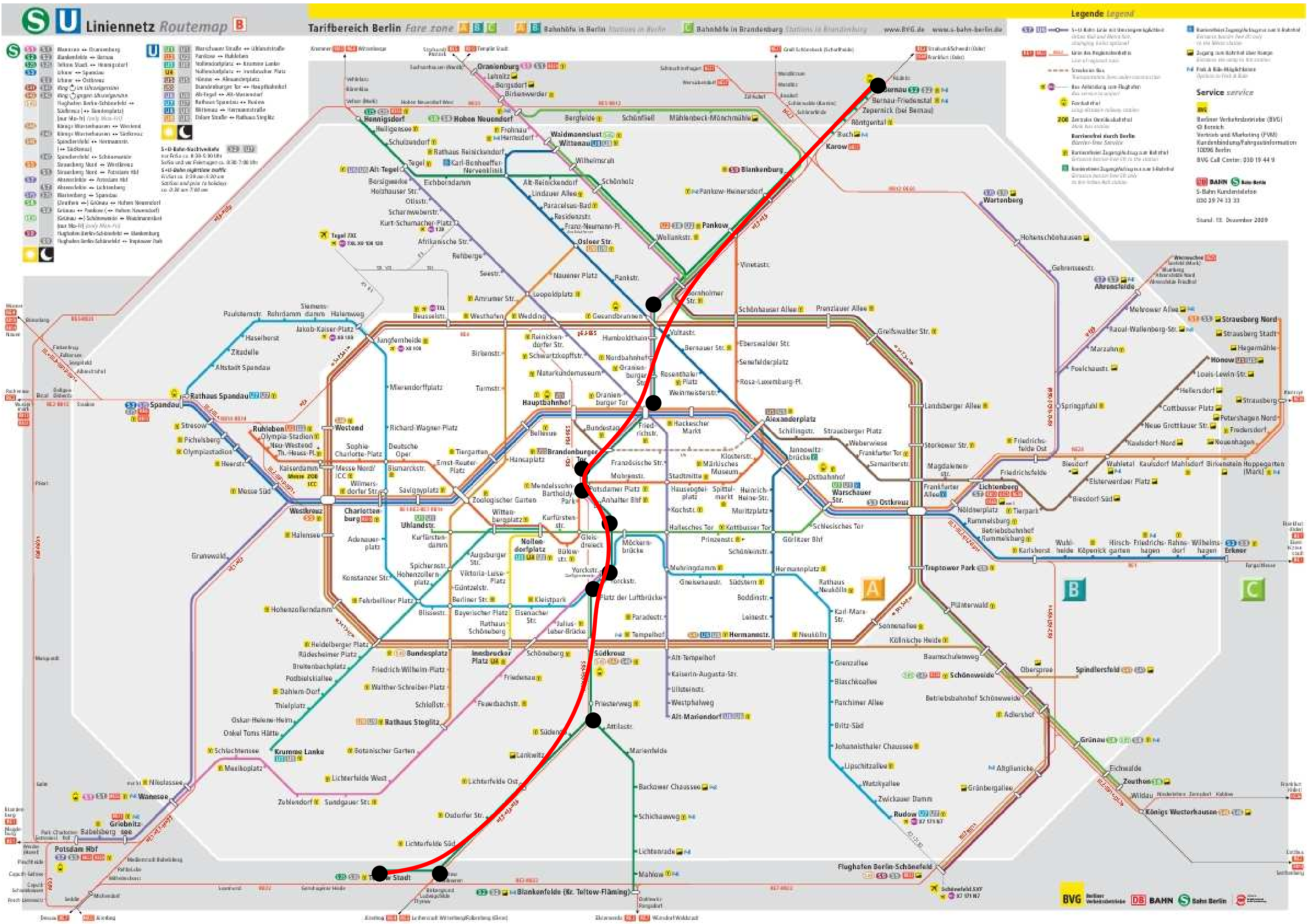
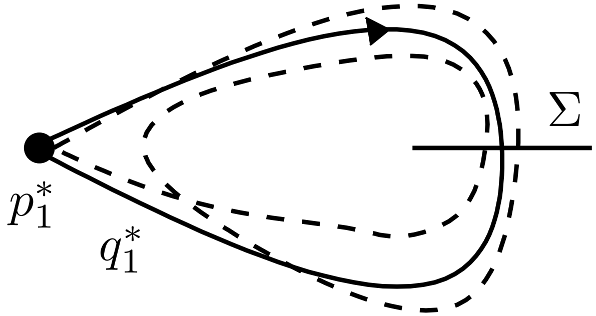


Schematics



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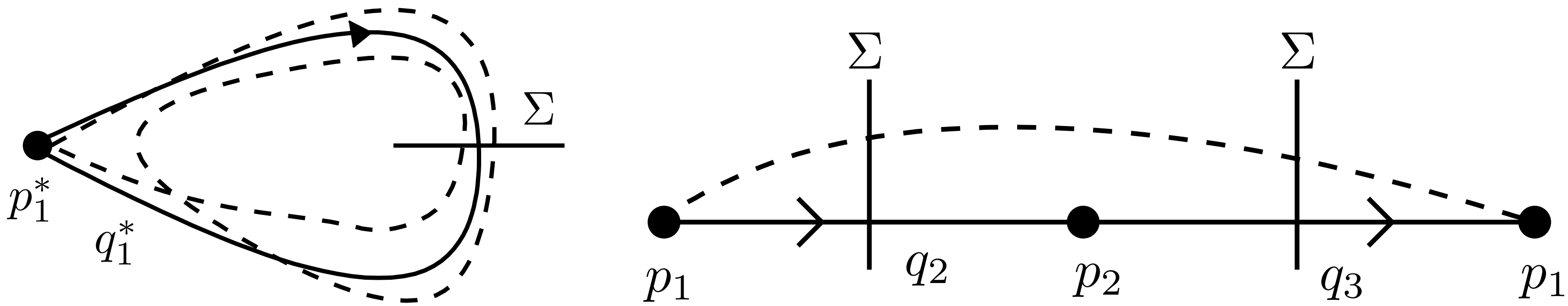

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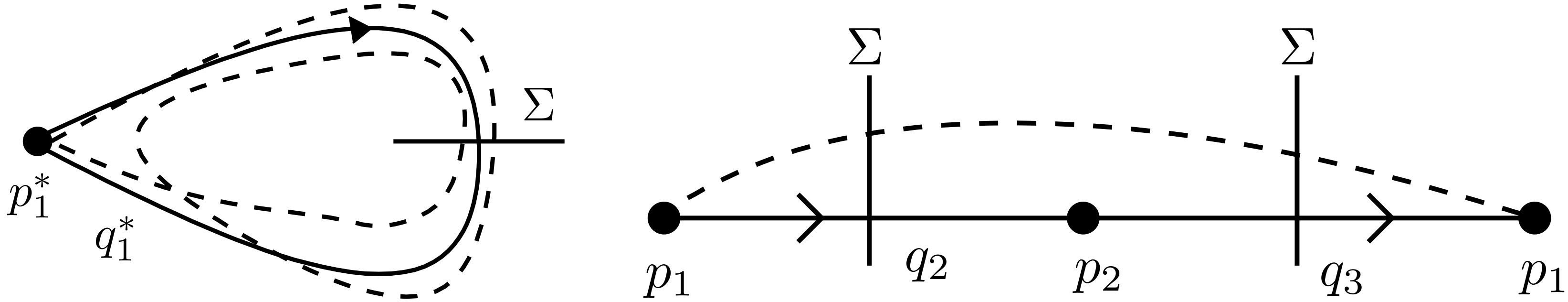

Approach $\approx$ Lyapunov-Schmidt reduction of matching pieces at Σ's

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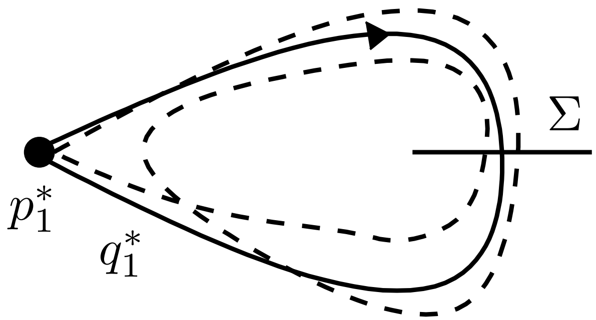
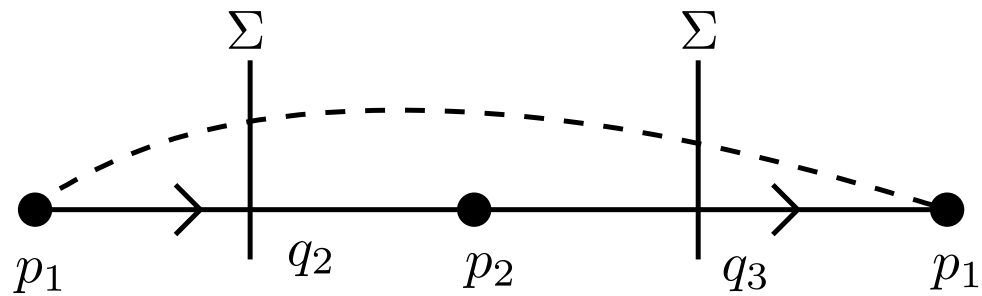
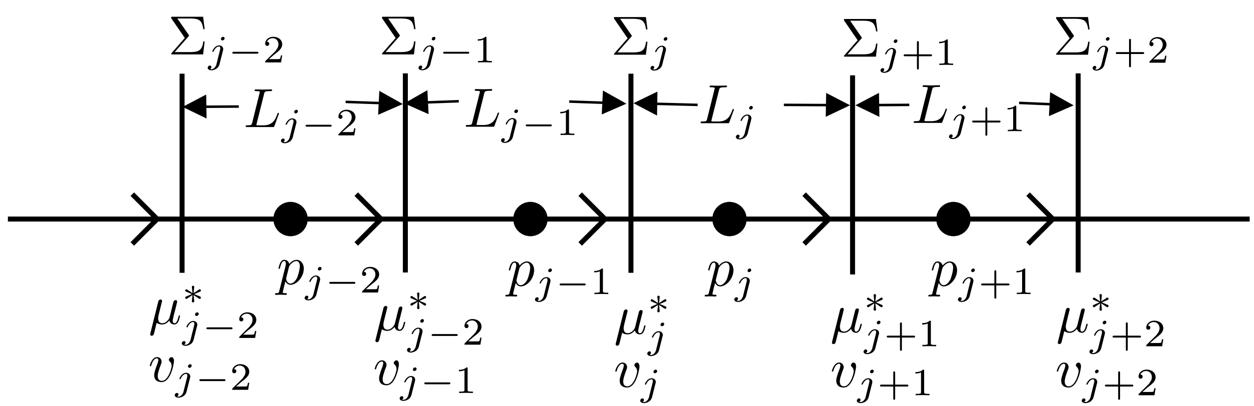




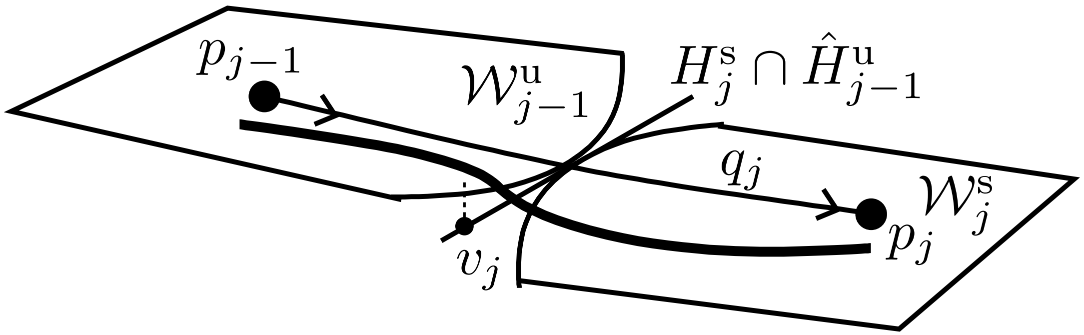
Approach $\approx$ Lyapunov-Schmidt reduction of matching pieces at Σ's

Result: algebraic equations that couple μ to 'geometric characteristics':



Tangency and assumptions

‘parameter’ v_j appears in case of tangency:



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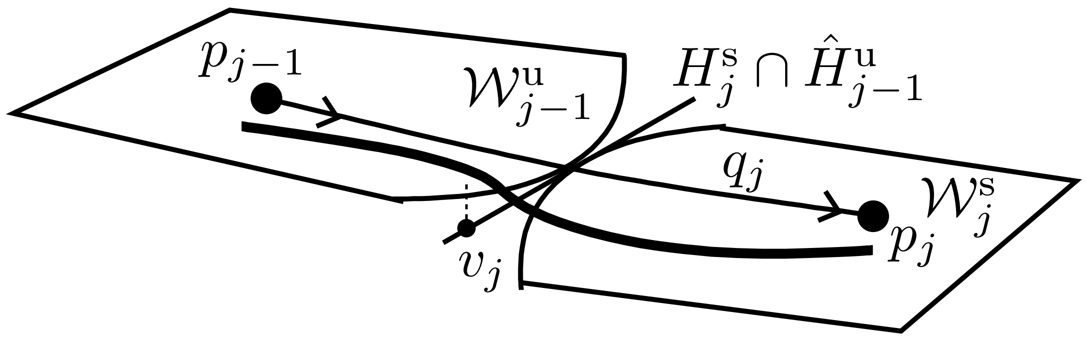

Also have v_j if set of heteroclinic point is higher than 1D.

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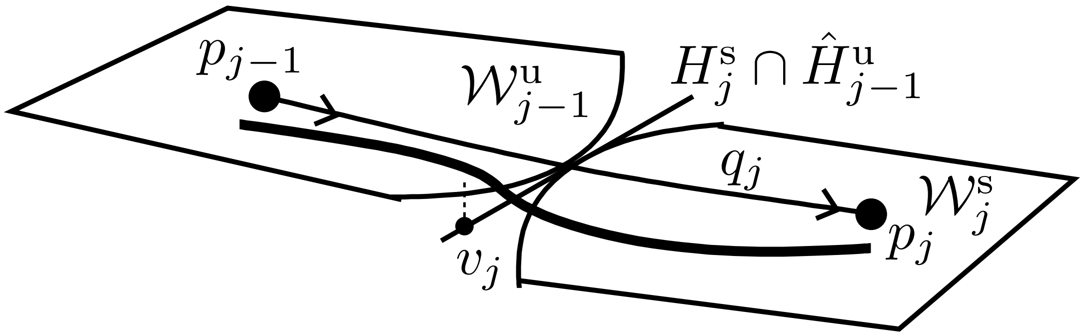

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Assumptions:

- parameters unfold generically

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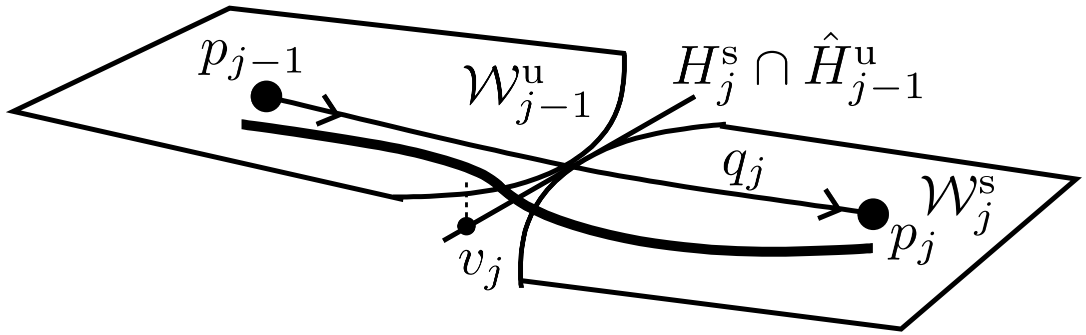


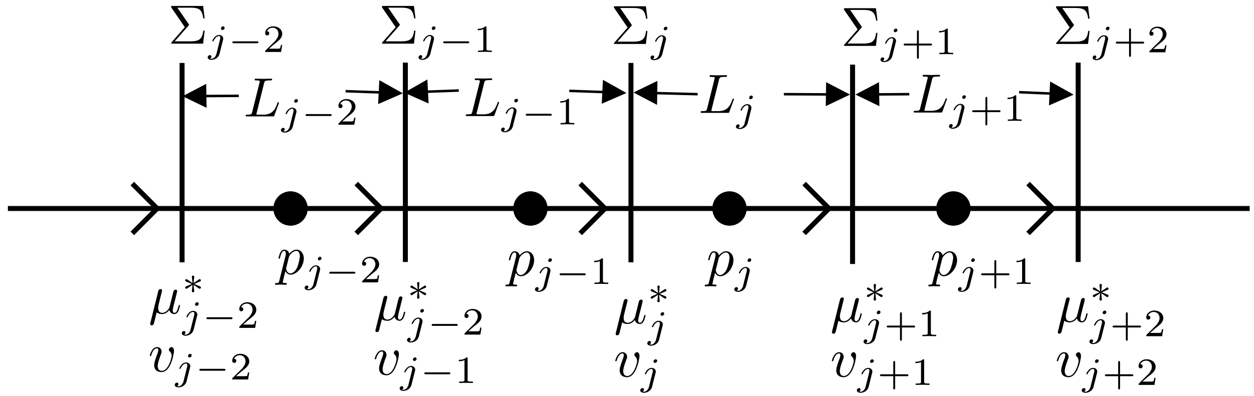
Also have v_j if set of heteroclinic point is higher than 1D.

Assumptions:

- parameters unfold generically
- p_j are hyperbolic with leading un/stable Floquet exponents

$$\nu_j^{\mathrm{u/s}} = \pm\kappa_j^{\mathrm{u/s}} + \mathrm{i}\sigma_j^{\mathrm{u/s}}$$

Rigorous matching: bifurcation equations



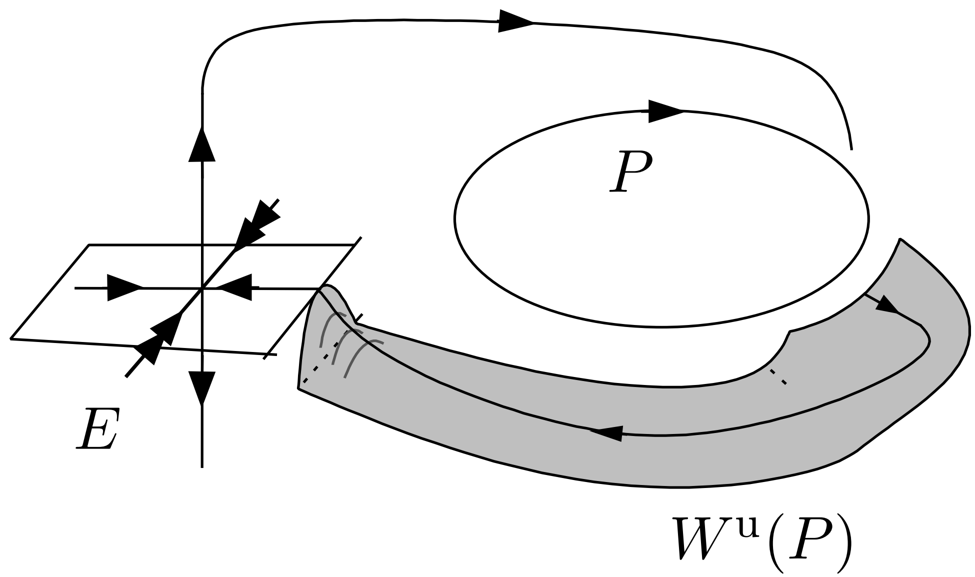
$$\begin{aligned}\mu_j^* \quad &= \mathcal{T}_j(v_j) + \mathrm{e}^{-2\kappa_j^{\mathrm{u}} L_j}\,\mathrm{Cos}(2\sigma_j^{\mathrm{u}} L_j + \beta_j^*(v_{j+1}, L_{j+1}))\zeta_j^*(v_{j+1}, L_{j+1}) \\ &+\mathrm{e}^{-2\kappa^{\mathrm{s}}_{j-1} L_{j-1}}\,\mathrm{Cos}(2\sigma^{\mathrm{s}}_{j-1} L_{j-1} + \gamma_j^*(v_{j-1}, L_{j-2})\xi_j^*(v_{j-1}, L_{j-2}) + \mathcal{R}_j.\end{aligned}$$

Solvability condition for repeatedly visited q_j: $\mu_j^* = \mu_{j'}^*$ if $q_j = q_{j'}$.

Can, e.g., reprove the well-known homoclinic bifurcation results,

and that a homoclinic orbit to a periodic orbit implies ‘chaos’.

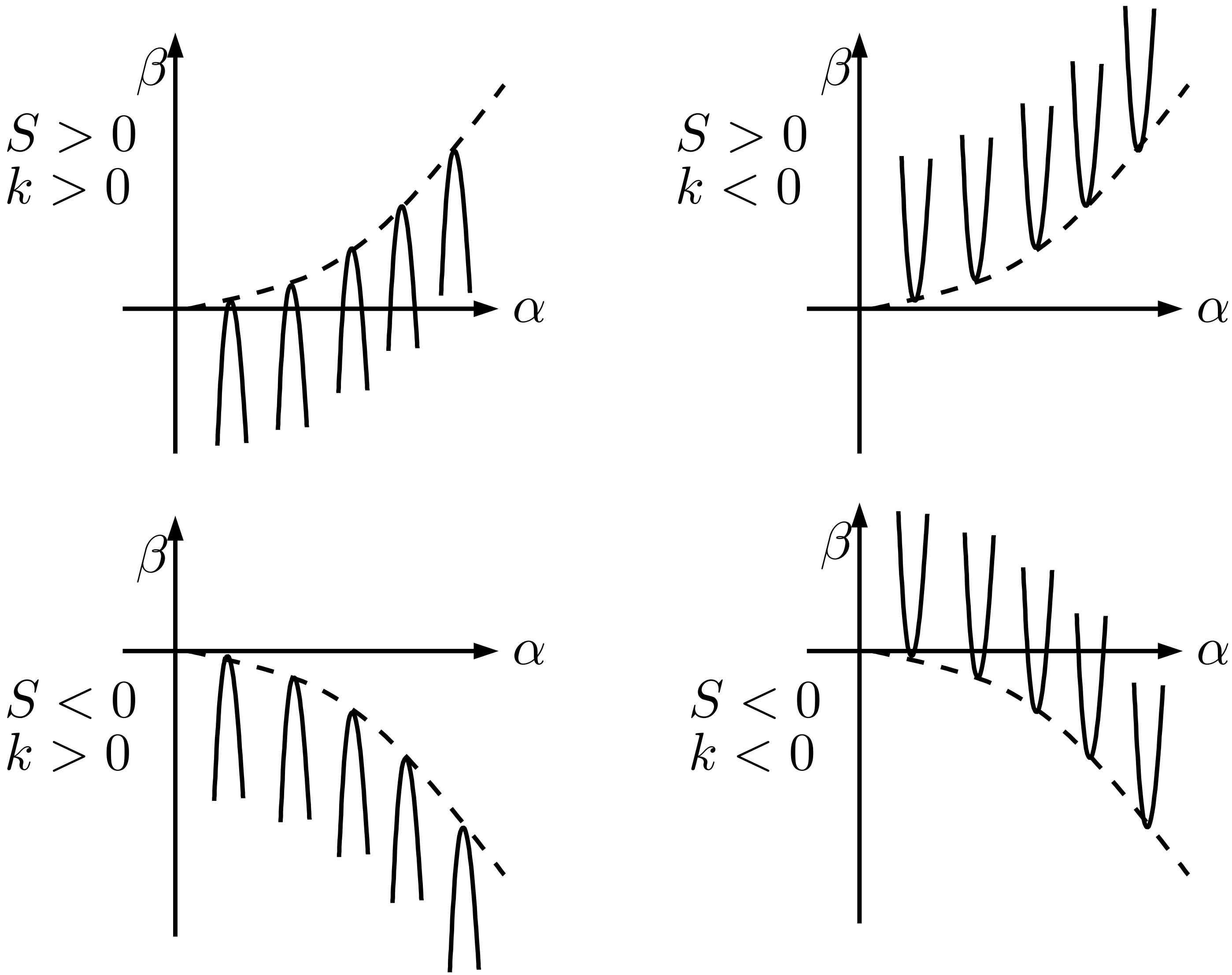
EP1t-cycle



Connection from P to E is tangent, connection from E to P has linear codimension $1 \Rightarrow$ need two parameters for unfolding.

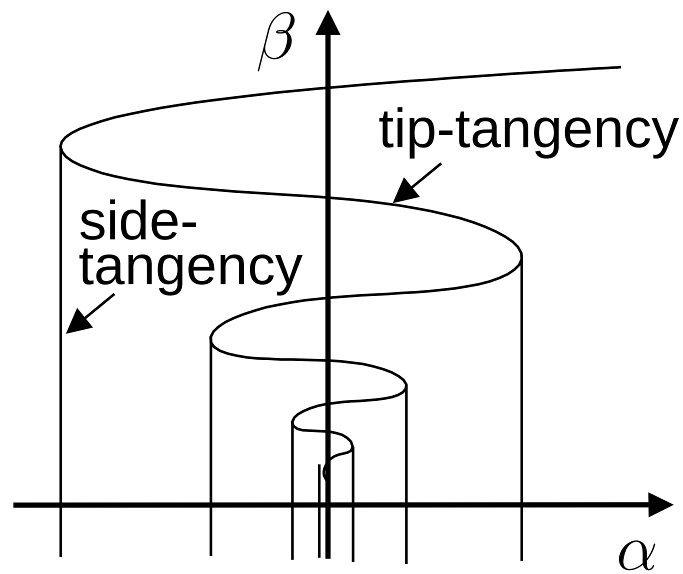
Sample bifurcation set: E-hom

Positive Floquet multipliers at P: E-hom occur on parabolas:



P-hom tangencies

Complex leading eigenvalues at E: P-hom tangencies occur on:



References:

- rigorous general reduction to bifurcation equations: R., to appear JDE

- partly formal for EP1t cycle: Champneys, Kirk, Knobloch, Oldeman, R., SIADS 8 (2009) 1261-1304

Topic 3

Transition to chaos in the wake of predator invasion

joint work with M. Smith (Cambridge), J. Sherratt (Edinburgh)

Topic 4

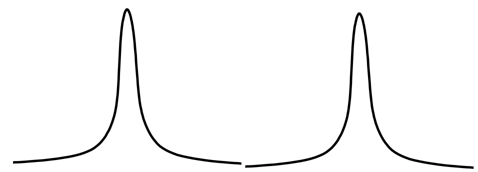
First and second order semi-strong interaction

partly joint work with J. Ehrt, M. Wolfrum (Berlin)

Weak interaction of pulses

Pulses at x_j, $j = 1, \ldots, N$,

initially with $x_j - x_i \geq L$, $L \gg 1$



Interaction through exponentially small superposition error in ‘tails’:

relative motion is exponentially slow,

$$\frac{\mathrm{d}}{\mathrm{d}t}(x_1 - x_2) = \mathcal{O}(\exp(-\kappa L)), \quad \kappa > 0 \text{ from tail geometry}$$

For N pulses, general rigorous theory [Zelik, Mielke ’07, ’09]:

Centre-manifold reduction to N-dimensional ‘pulse manifold’ and ODE

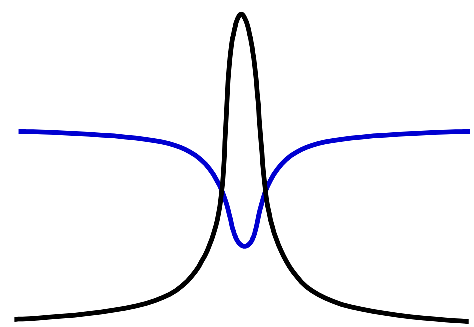
for pulse motion.

Diffusion length

For $\varepsilon = 1/L$, $z = \varepsilon x$ obtain

$$u_t = \varepsilon^2 D u_{zz} + f(u; \mu).$$

Weak interaction occurs when diffusion lengths of all components are of the same order in ε as $\varepsilon \to 0$.



Semi-strong interaction

Semi-strong interaction occurs if some diffusion lengths are order one:

$$\begin{aligned}\partial_t u &= D_u \partial_{xx} u + F(u,v) \\ \partial_t v &= \varepsilon^2 D_v \partial_{xx} v + G(u,v),\end{aligned}$$

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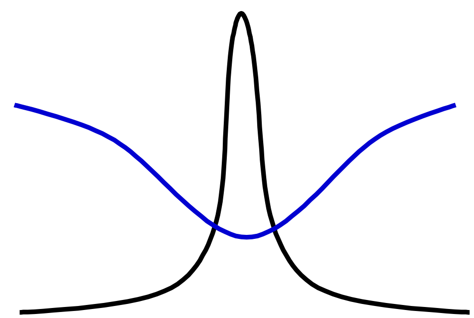
Pulse: solution that is localised in v as $\varepsilon \to 0$.



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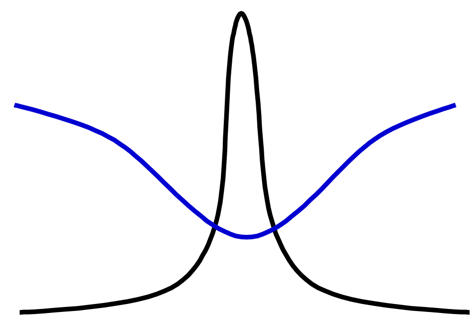


Reduction to pulse-manifold and interaction ODE so far only formal.

Nature of problem also depends on details of scaling:

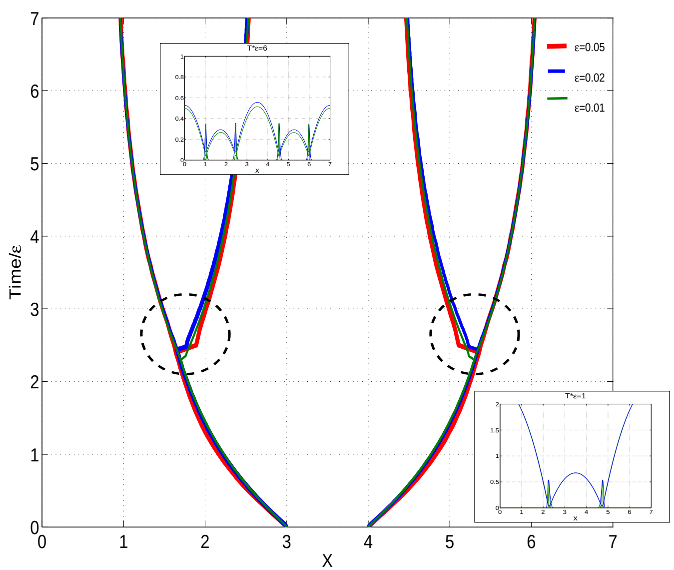
first vs. second order cases.

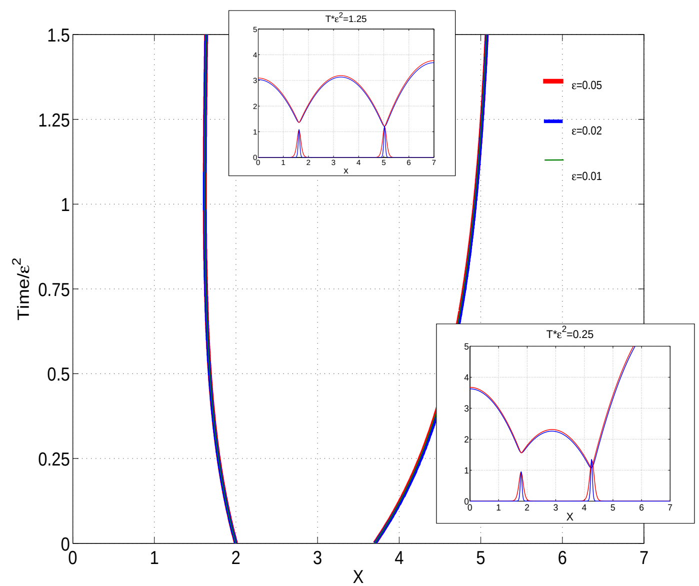
1st and 2nd order semi-strong interaction

Example:

Schnakenberg model

$$\begin{aligned} \partial_t u &= \partial_{xx} u + \alpha - uv^2 \\ \partial_t v &= \varepsilon^2 \partial_{xx} v + \beta - v + uv^2. \end{aligned}$$





1st order semi-strong:

velocity $c = \mathcal{O}(\varepsilon)$,

$\alpha = 0.9$, $\beta = 0.1$.

2nd order semi-strong:

velocity $c = \mathcal{O}(\varepsilon^2)$,

$\alpha = 1.3\sqrt{\varepsilon}$, $\beta = 0.1$.

Rescaled equations

Original:

$$\begin{aligned}\partial_t u &= \partial_{xx} u + \alpha - uv^2\\ \partial_t v &= \varepsilon^2 \partial_{xx} v + \beta - v + uv^2.\end{aligned}$$

1st order case: $\alpha = \hat{\alpha}$, $u = \hat{u}$, $v = \hat{v}/\varepsilon$

$$\begin{aligned}\partial_t \hat{u} &= \partial_{xx} \hat{u} + \hat{\alpha} - \tfrac{1}{\varepsilon^2}\hat{u}\hat{v}^2\\ \partial_t \hat{v} &= \varepsilon^2 \partial_{xx} \hat{v} + \varepsilon\hat{\beta} - \hat{v} + \tfrac{1}{\varepsilon}\hat{u}\hat{v}^2.\end{aligned}$$

2nd order case: $\alpha = \sqrt{\varepsilon}\check{\alpha}$, $u = \sqrt{\varepsilon}\check{u}$, $v = \check{v}/\sqrt{\varepsilon}$

$$\begin{aligned}\partial_t \check{u} &= \partial_{xx} \check{u} + \check{\alpha} - \tfrac{1}{\varepsilon}\check{u}\check{v}^2\\ \partial_t \check{v} &= \varepsilon^2 \partial_{xx} \check{v} + \sqrt{\varepsilon}\check{\beta} - \check{v} + \check{u}\check{v}^2.\end{aligned}$$

2nd order case

Reduction setup

Idea: expand $u = u_0 + \mathcal{O}(\varepsilon)$, $v = v_0 + \mathcal{O}(\varepsilon)$ and derive equations for u_0, v_0 assuming $\partial_t \check{v}_0 = \partial_t \check{u}_0 \equiv 0$.

$$
\begin{aligned}
\mathcal{O}(\varepsilon) &= \partial_{xx}\check{u} + \check{\alpha} - \frac{1}{\varepsilon}\check{u}\check{v}^2 \\
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Large scale: assume $\check{v}_0 \equiv 0$ away from pulse locations, and set $\varepsilon = 0$.

$$0 = \partial_{xx}\check{U}_0 + \check{\alpha} \quad \Rightarrow \quad \check{U}_0 = -\frac{\check{\alpha}}{2}x^2 + cx + d.$$

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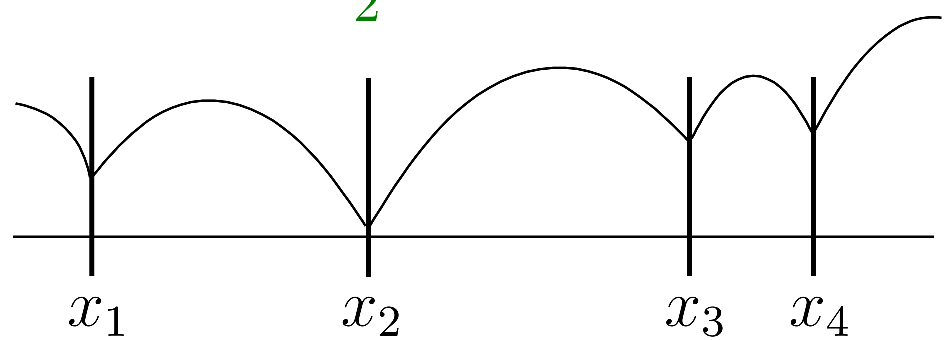
Piecewise smooth solution composed of parabolas.



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$$\begin{aligned} \partial_\xi \check{u}_0 &= 0 \\ \partial_\xi \check{p}_0 &= \check{u}_0 \check{v}_0^2 \quad (\check{p}_0 = \partial_x \check{u}_0) \end{aligned} \qquad \begin{aligned} \partial_{\xi\xi} \check{v}_0 &= \check{v}_0 - \check{u}_0 \check{v}_0^2 . \end{aligned}$$

Only one 2nd order ODE!

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Now construct (leading order) solutions by matching large and small scale solutions: yields algebraic equations.

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Now construct (leading order) solutions by matching large and small scale solutions: yields algebraic equations.

Substitution and projection gives the reduced dynamics

$$\frac{\mathrm{d}}{\mathrm{d}t} x_j(t) = 2\varepsilon^2 \frac{b_j}{a_j} + \mathcal{O}(\varepsilon^3)$$

in accordance with [Doelman, Kaper ’03; Ward et al ’05].

Transverse stability

General transverse stability of pulse manifold?

Leading order eigenvalue problem:

$$L(\check{u}_0, \check{v}_0) := \begin{pmatrix} \partial_{xx} - \frac{1}{\varepsilon}\check{v}_0^2 & \frac{2}{\varepsilon}\check{u}_0\check{v}_0 \\ \check{v}_0^2 & \varepsilon^2\partial_{xx} - 1 + 2\check{u}_0\check{v}_0 \end{pmatrix},$$

$$L(\check{u}_0, \check{v}_0)\begin{pmatrix} \Phi \\ \Psi \end{pmatrix} = \lambda \begin{pmatrix} \Phi \\ \Psi \end{pmatrix}.$$

Large scale: set $\check{v}_0 = 0$ and then $\varepsilon = 0$:

$$\partial_{xx}\Phi = \lambda\Phi, \quad (1 + \lambda)\Psi = 0.$$

Small scale and matching

For 1-pulse and Neumann b.c. get inhomogeneous linear ODE

$$\partial_{\xi\xi}\psi = (1 - 2a_1\check{v}_0(\xi) - \lambda)\psi(\xi) - \Phi(x_1)\check{v}_0^2(\xi).$$

Constraint from matching:

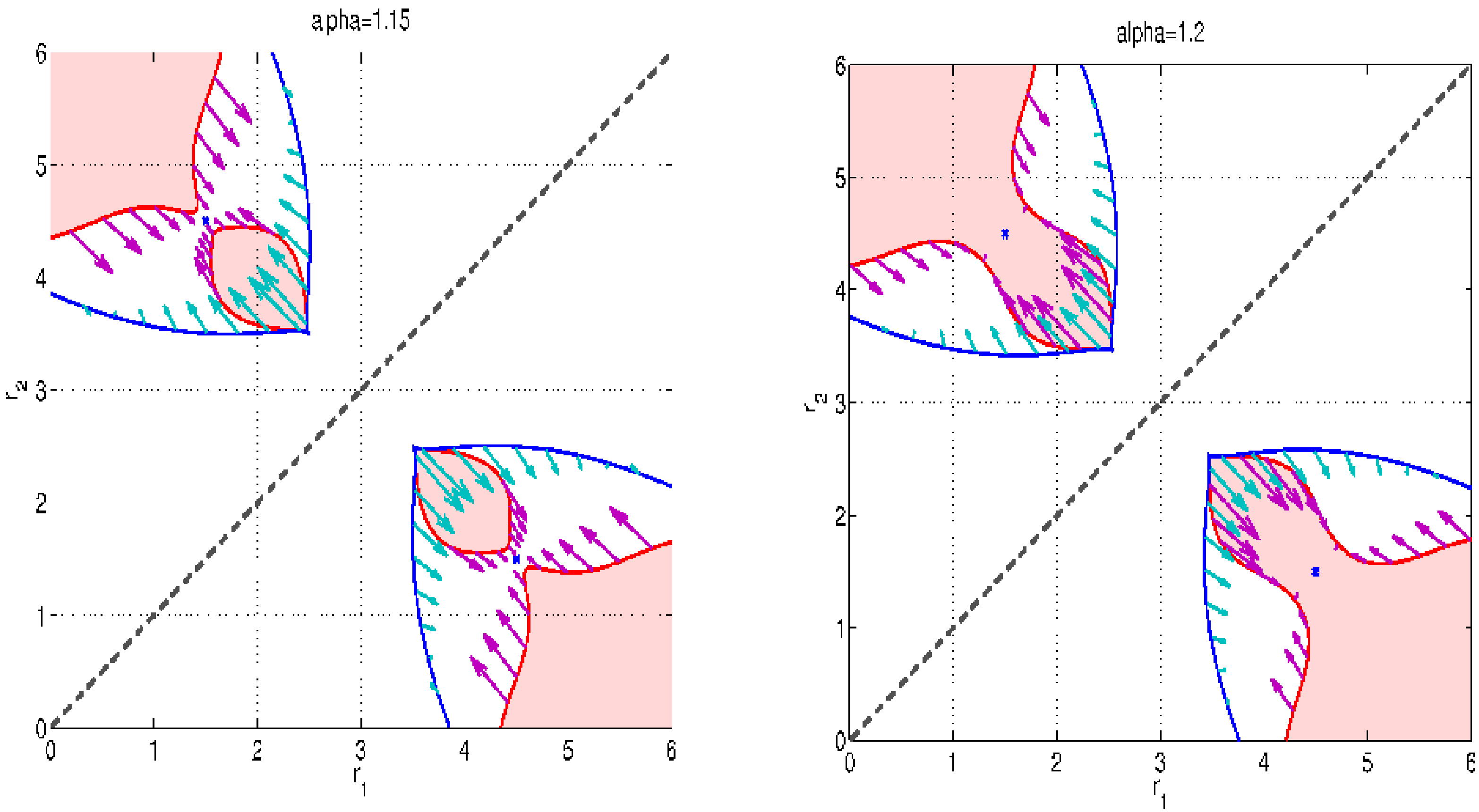
$$\int_{-\infty}^{\infty} 3\left(1 - \tanh^2(\xi/2)\right)\psi(\xi)d\xi =$$

$$\Phi(x_1)\left(\sqrt{\lambda}\left(\tanh\sqrt{\lambda}(x_1 - L) - \tanh\sqrt{\lambda}\,x_1\right) - \frac{6}{a_1^2}\right).$$

NLEP method [Doelman, Gardner, Kaper '01] would express solutions in terms of hypergeometric functions.

Instead, we use numerical continuation to compute stability boundary.

Stability boundary for Neumann 2-pulse



PDE numerics when crossing boundary: annihilation (‘overcrowding’)

Numerics delicate: delayed Hopf-bifurcation in PDE!

1st order case:

pulse motion of order ε

Reduction setup

Expand $u = u_0 + \varepsilon u_1 + \mathcal{O}(\varepsilon^2)$, $v = v_0 + \mathcal{O}(\varepsilon)$

[...]

Reduction setup

Expand $u = u_0 + \varepsilon u_1 + \mathcal{O}(\varepsilon^2)$, $v = v_0 + \mathcal{O}(\varepsilon)$

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Same as 2nd order case: piecewise smooth solution of parabolas.

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Same as 2nd order case: piecewise smooth solution of parabolas.

Small scale problem ($\xi = (x - x_j)/\varepsilon$):

$$\begin{aligned}
\partial_{\xi\xi}\hat{u}_1 &= \hat{u}_1\hat{v}_0^2 \\
\partial_{\xi\xi}\hat{v}_0 &= c\partial_\xi\hat{v}_0 + \hat{v}_0 - \hat{u}_1\hat{v}_0^2.
\end{aligned}$$

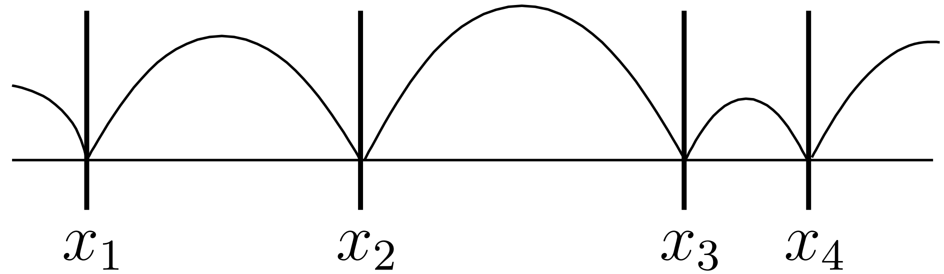
In contrast to first order case: two coupled 2nd order ODE!

Matching

Matching to large scale requires

$$\hat{U}_0(x_j) = 0 \ (!)$$

$$\partial_\xi \hat{u}_1(\pm\infty) = \partial_x \hat{U}_0(x_j \pm 0)$$



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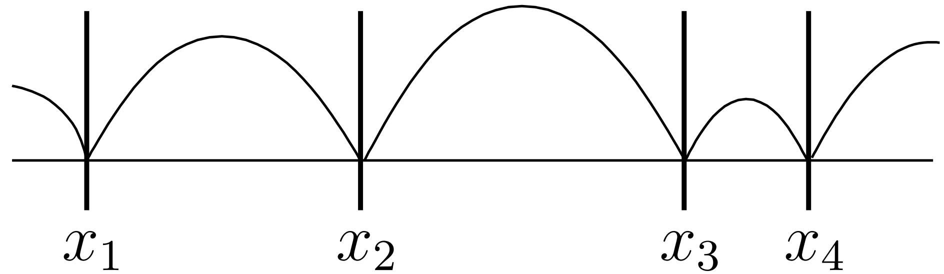
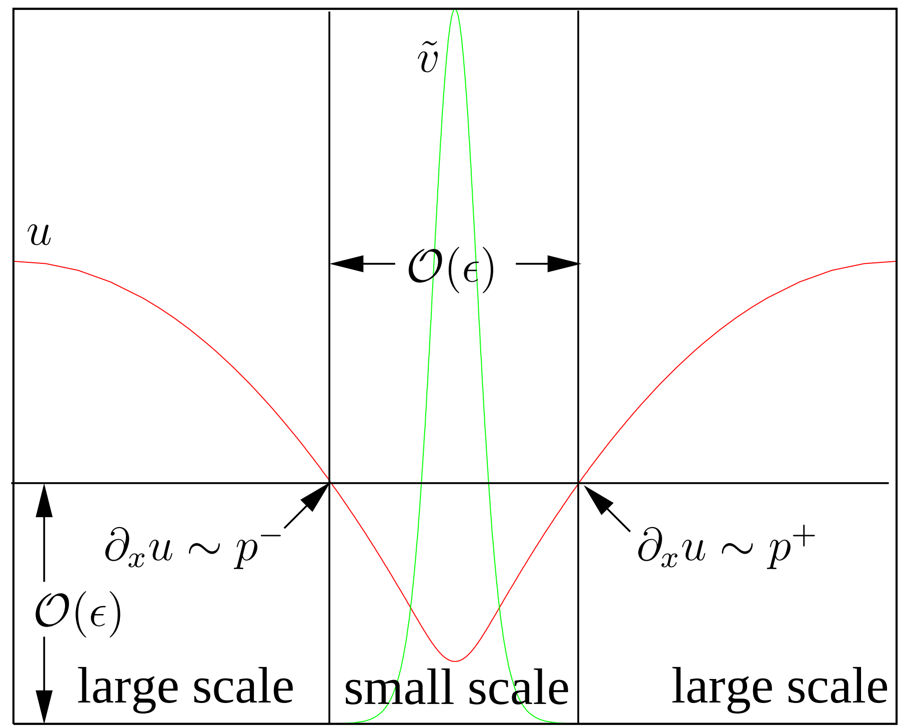




Existence problem only local:

nearest neighbor coupling

$\Rightarrow$ parameters $p_\pm := \partial_x \hat{U}_0(x_j \pm 0)$

Transverse stability for 1st order

Large scale: $\partial_{xx}\Phi_0 = \lambda\Phi_0, \quad (1+\lambda)\Psi_0 = 0$ (same as 2nd order).

Small scale:

($\hat{\psi} = \varepsilon\psi$)

$$\begin{aligned}
\partial_{\xi\xi}\phi_0 - \hat{v}_0^2\phi_0 + 2\hat{u}_1\hat{v}_0\hat{\psi}_0 &= 0 \\
\partial_{\xi\xi}\hat{\psi}_0 + \hat{v}_0^2\phi_0 - \hat{\psi}_0 + 2\hat{u}_1\hat{v}_0\hat{\psi}_0 &= \lambda\hat{\psi}_0,
\end{aligned}$$

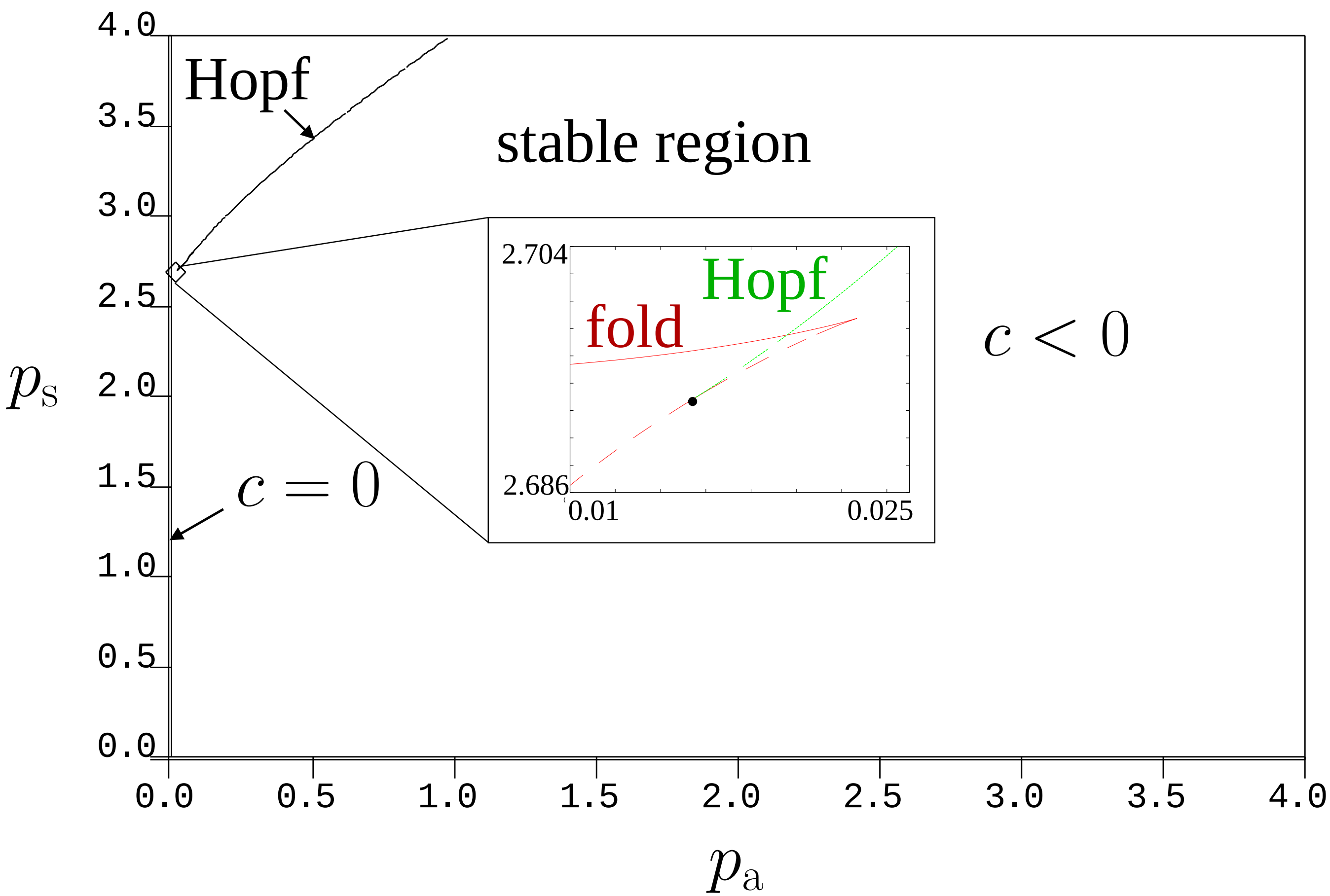
$$\phi_0(-\infty) = 1,\ \hat{\psi}(\pm\infty) = 0,\ \partial_\xi\phi_0(\pm\infty) = 0.$$

Small and large scale problems decouple:

as existence, stability determined locally!

Universal existence and stability map

We solve this as a boundary value problem again by continuation:



PDE dynamics when crossing is pulse-splitting, again with delay effect.

Outlook

- Manuscript 1 submitted, manuscript 2 to be submitted...

- Rigorous reduction to pulse manifolds as in weak interaction?

[Approximate reduction: Promislow, Kaper, Doelman, van Heijster]

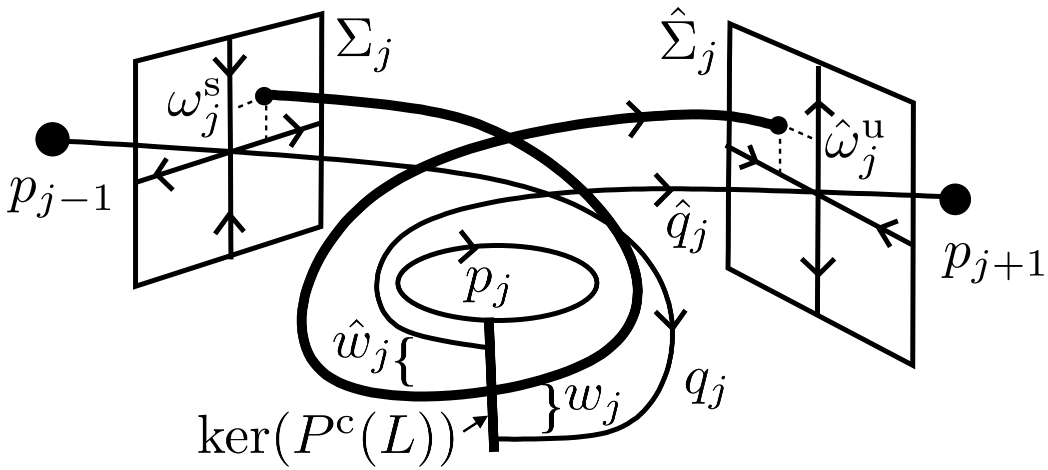
- Analysis of delayed bifurcations?

The end.

Coordinates for orbit segments that pass p_j

Here 'hat' for outflow from p_j, e.g., $\hat{q}_j = q_{j+1}$.

Decompose neighbourhood of $q_j \cup \hat{q}_j$ by un/stable directions inherited from p_k, $k = j-1, j, j+1$ via exponential trichotomies.



$\Rightarrow$ orbits parametrised by un/stable coordinates at $\hat{q}_j(0)$ / $q_j(0)$: $\hat{\omega}_j^{\mathrm{u}}$ / ω_j^{s}.

Proof: Contraction for var-o-const operator in suitable function space.