Ecological Motivation

Predator-prey interaction can cause spatio-temporally cyclic population dynamics. For instance for vole-weasel interaction in Fennoscandia [Ranta, Kaitala 1997, Nature].

vole

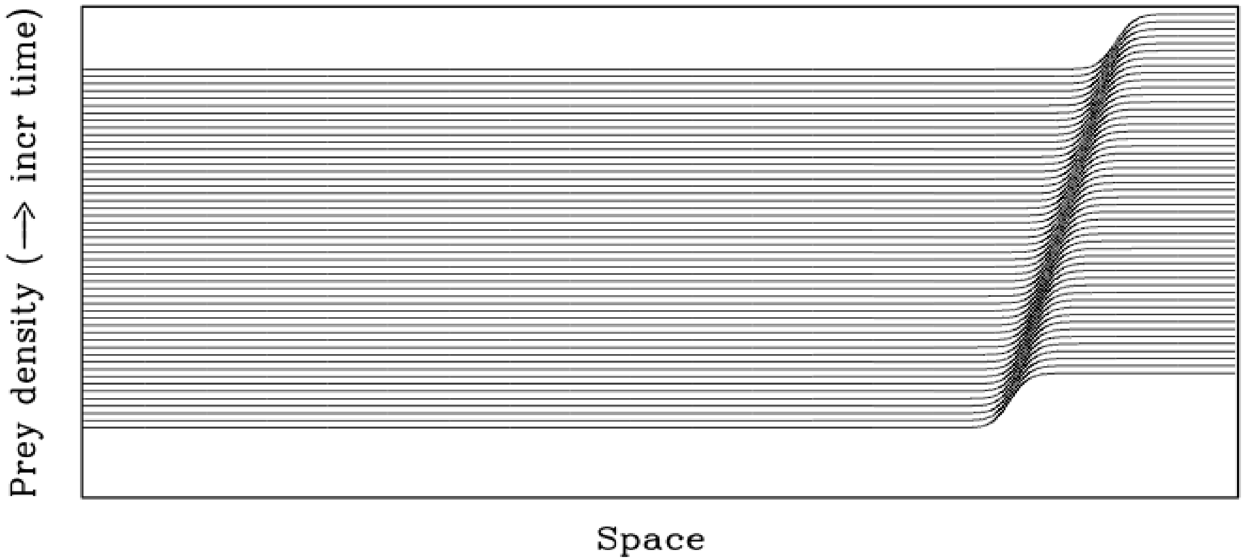
weasel

Invasion: predator invades region dominated by prey.

Invasion, and its wake

Unstable prey-only,
stable co-existence:

Invasion of steady
co-existence.



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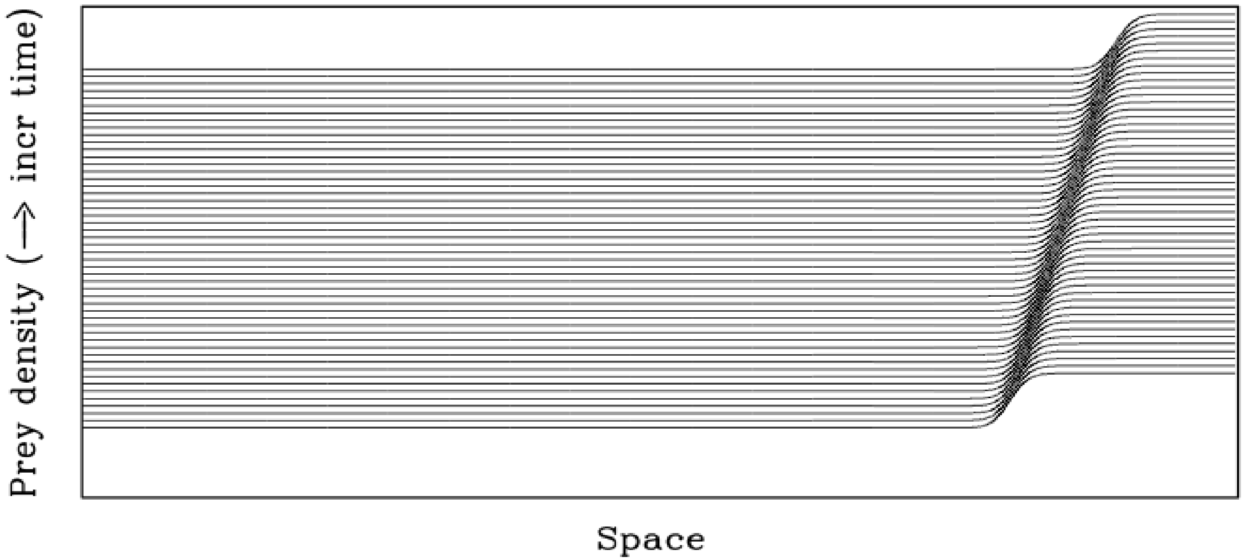
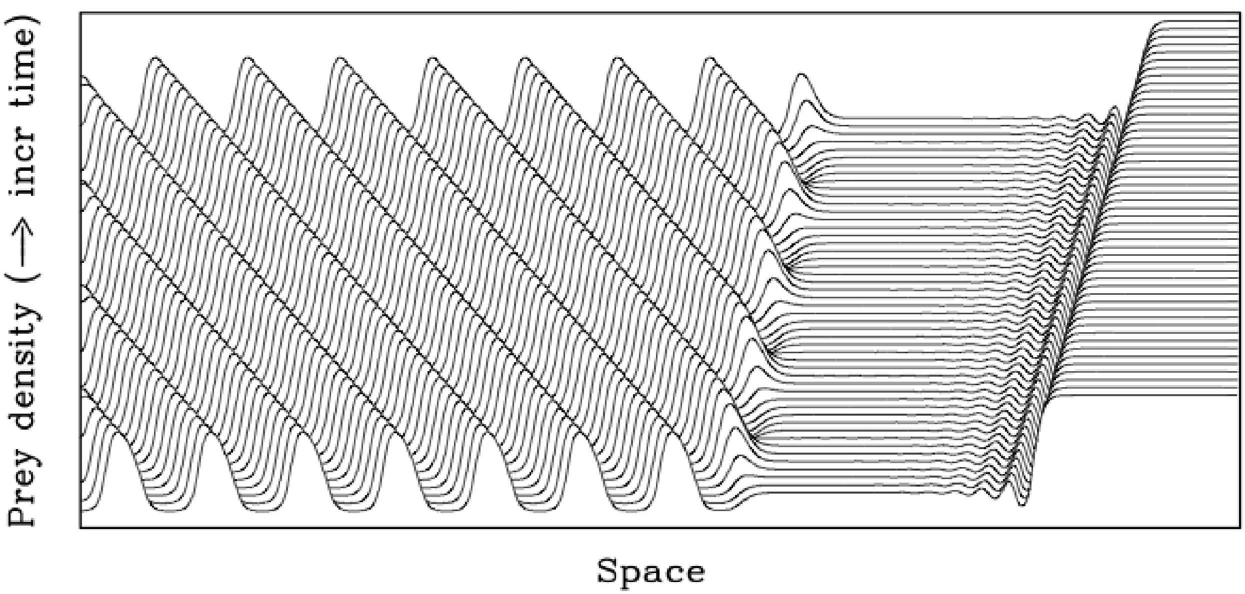


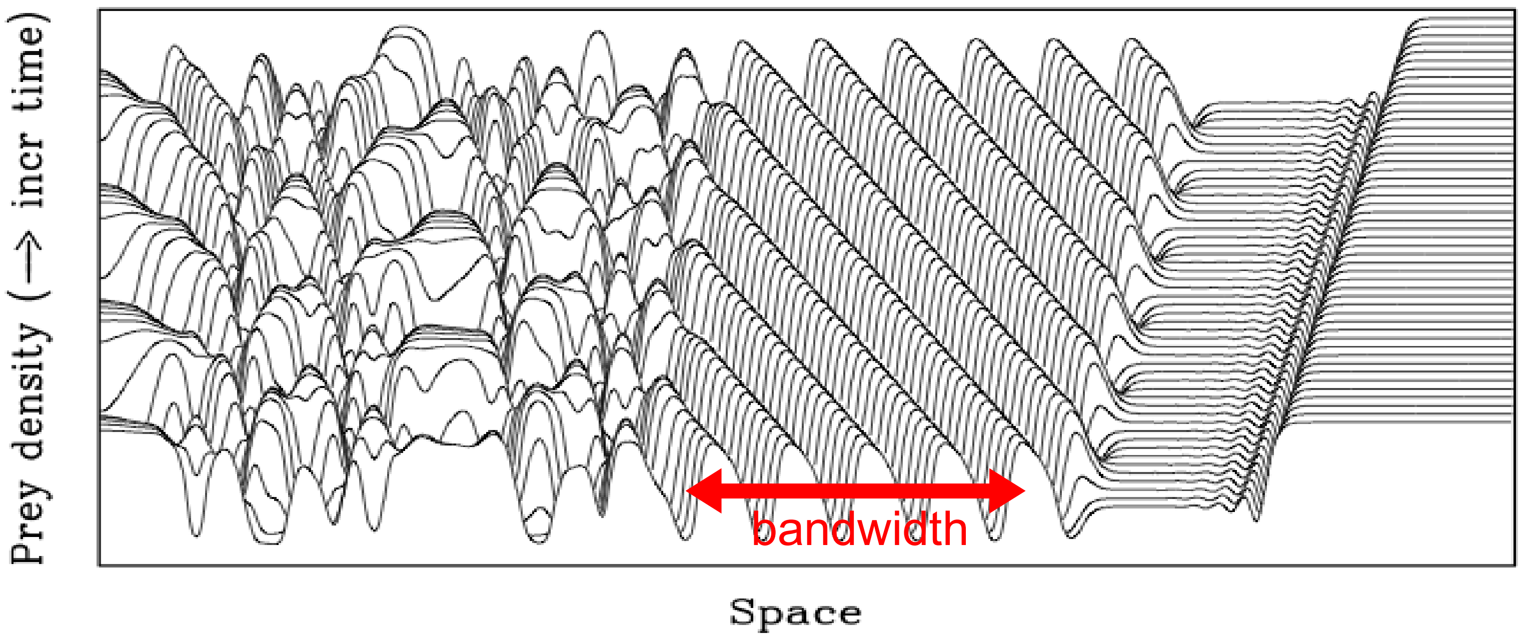
Unstable prey-only,
unstable co-existence,
and stable oscillations
in form of a `wave train':

Invasion of space-time
periodic co-existence.



Chaotic wake and bandwidth

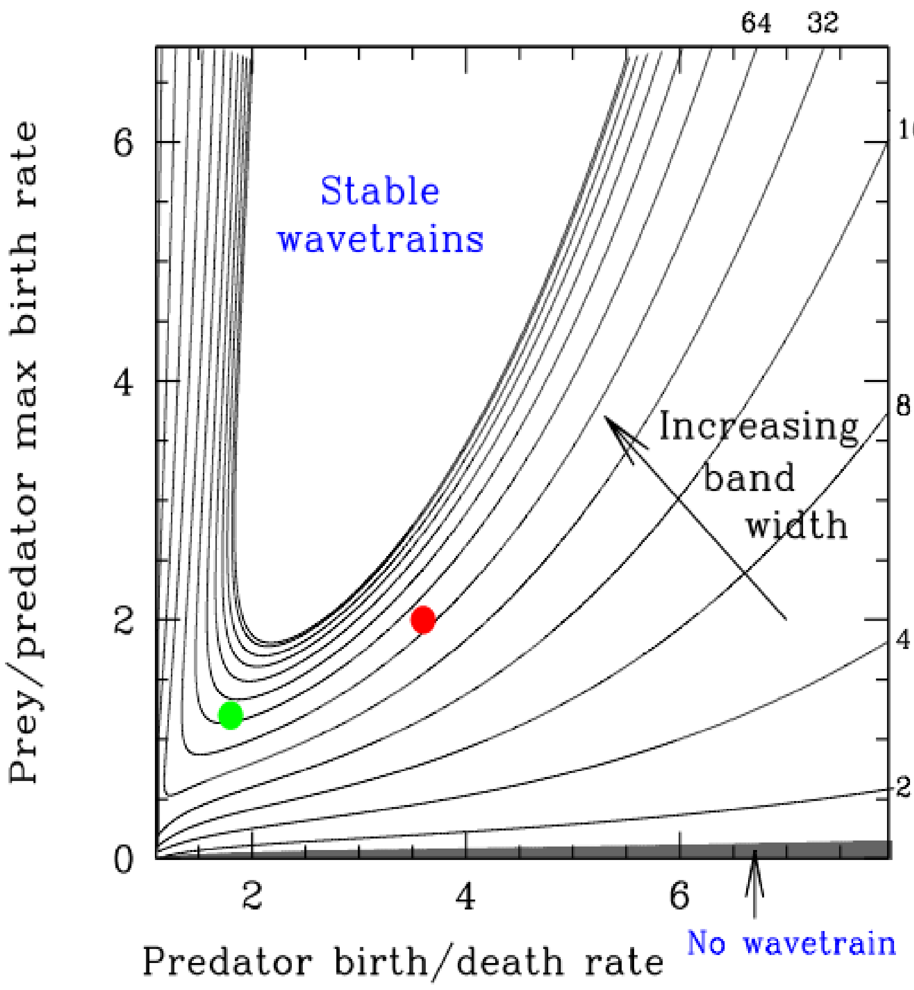
If the wave train in the wake is also unstable: invasion of `chaos'.



Bandwidth: `extend of region with regular wave train oscillations.'

Measures degree of regularity despite instability: unstable wave train can be prevalent in application, if domains size smaller than bandwidth.

Bandwidth landscape

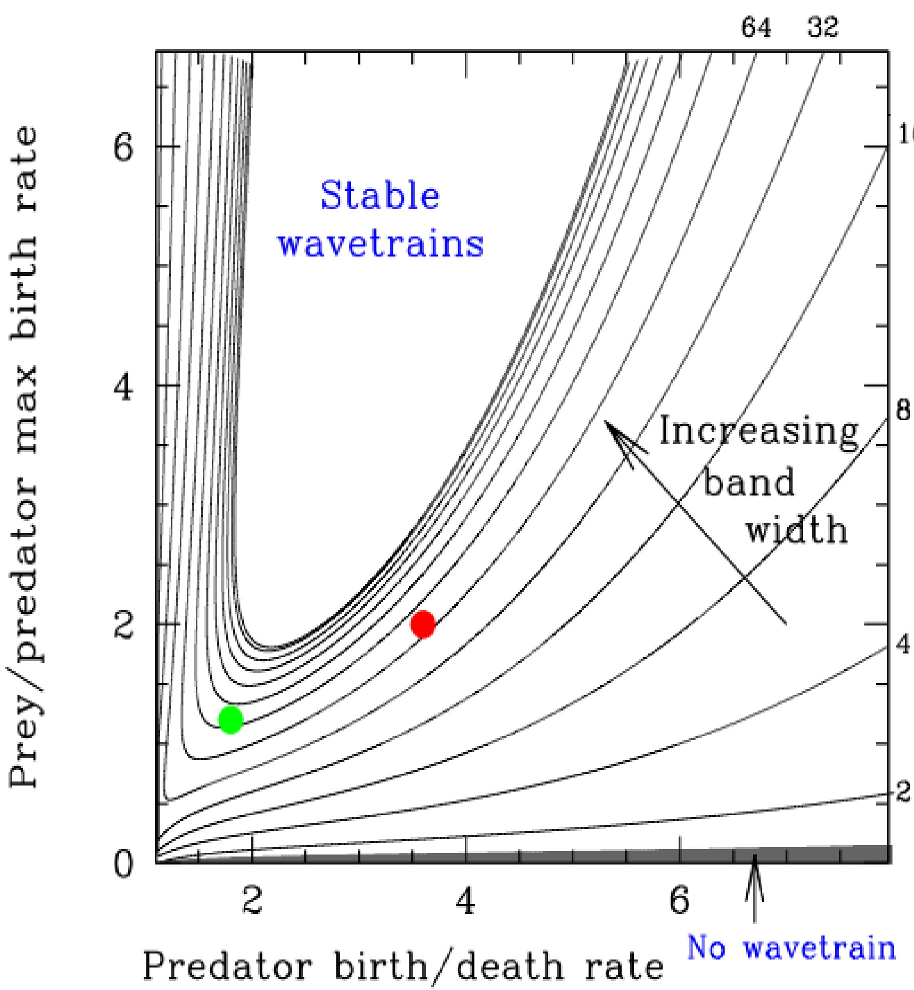


- Weasel-vole parameters:

  5% increase in vole birthrate implies 22% increase in bandwidth

(Rosenzweig-MacArthur model)

Bandwidth landscape



- Weasel-vole parameters:

  5% increase in vole birthrate implies 22% increase in bandwidth

- plankton parameters:

  5.2% increase in plankton birthrate implies doubling of bandwidth

It is known that climate change affects parameters.
Our results suggest significant impact on spatio-temporal in/coherence.

(Rosenzweig-MacArthur model)

Complex Ginzburg-Landau equation

For simplicity we consider the λ-ω system

$$\begin{aligned}
\partial_t u &= \partial_{xx} u + (1 - r^2)u - (\omega_0 - \omega_1 r^2)v \\
\partial_t v &= \partial_{xx} v + (1 - r^2)v - (\omega_0 - \omega_1 r^2)u, \\
r &= \sqrt{u^2 + v^2}
\end{aligned}$$

Complex Ginzburg-Landau equation

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$$\begin{aligned}\partial_t u &= \partial_{xx} u + (1 - r^2)u - (\omega_0 - \omega_1 r^2)v \\ \partial_t v &= \partial_{xx} v + (1 - r^2)v - (\omega_0 - \omega_1 r^2)u, \\ r &= \sqrt{u^2 + v^2}\end{aligned}$$

Which is equivalent to CGL with real diffusion (`dispersionless')

$$\partial_t u = \partial_{xx} u + u - (1 + \beta \mathrm{i})|u|^2 u$$

It is the normal form for reaction-diffusion systems near a supercritical Hopf-bifurcation of the reaction kinetics. Wave trains are explicitly:

$$u(x,t) = R\exp[\mathrm{i}(\omega t - kx)],\ \omega = -\beta R^2,\ k^2 = 1 - R^2$$

Stability and spectrum

Linearising the PDE in a wave train yields `dispersion relation' for temporal and spatial modes

$$d(\lambda, \nu) = 0,\ \lambda, \nu \in \mathbb{C}$$

Analogous to characteristic equation, e.g., $\exp(\lambda t + \nu x)$ in

$$\partial_t u = \partial_{xx} u + c\partial_x u + au \to \lambda \quad = \nu^2 + c\nu + a$$

Stability and spectrum

Linearising the PDE in a wave train yields `dispersion relation' for (complex!) temporal and spatial modes

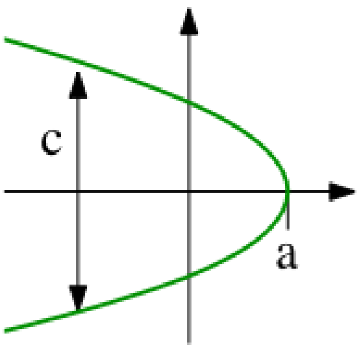
$$d(\lambda,\nu)=0,\ \lambda,\nu\in\mathbb{C}$$

Analogous to characteristic equation, e.g., $\exp(\lambda t+\nu x)$ in

$$\partial_t u=\partial_{xx}u+c\partial_x u+au\to\lambda\ \ =\nu^2+c\nu+a$$

Spectrum for $\nu=\mathrm{i}\kappa,\ \kappa\in\mathbb{R}$

(as in Fourier transform)



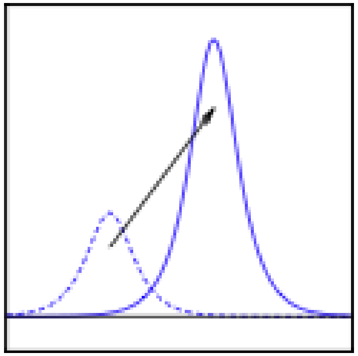
$\mathrm{Re}(\lambda(i\kappa))>0$ exponential growth of perturbations.

→instability also of nonlinear PDE.

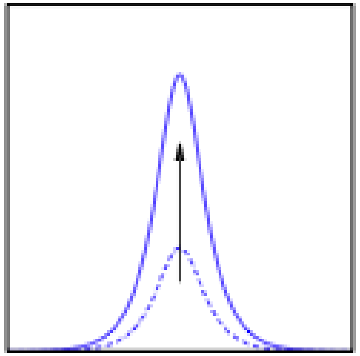
Absolute and convective instability

Instabilities in space-time:
growth in norm vs. pointwise growth

[Deissler; Brevdo, Bridges; van Saarloos; Sandstede, Scheel; R. ...]



convective



absolute

Absolute and convective instability

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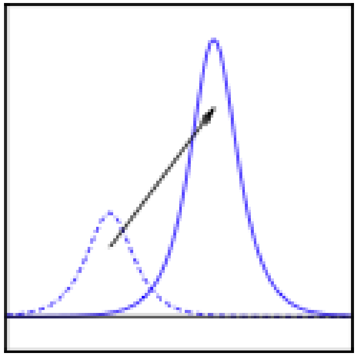
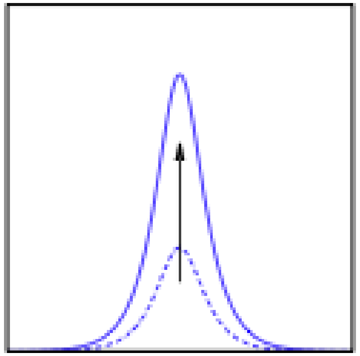


convective



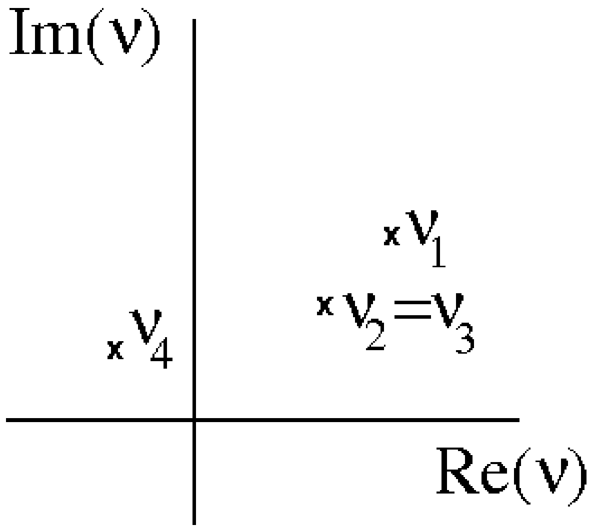
absolute

Co-moving frame $y = x - Vt$: change convective ↔ absolute

Absolute instability and branch points

Theory: absolute instability if there is unstable double root of the dispersion relation satisfying

$$
\begin{aligned}
d(\lambda,\nu_j;V) &= \partial_\nu d(\lambda,\nu_j;V)=0\\
\mathrm{Re}(\nu_j) &\geq \mathrm{Re}(\nu_{j+1})\\
j &= 1,\ldots,3\\
\boxed{\nu_2 \;=\; \nu_3}
\end{aligned}
$$



Denote most unstable by $\lambda_{\max}(V),\ \nu_{\max}(V)$

Bandwidth coefficient

Bandwidth formula $-\log(\mathcal{F})/\mathrm{Re}[\lambda'_{\mathrm{max}}(V_{\mathrm{band}})]$

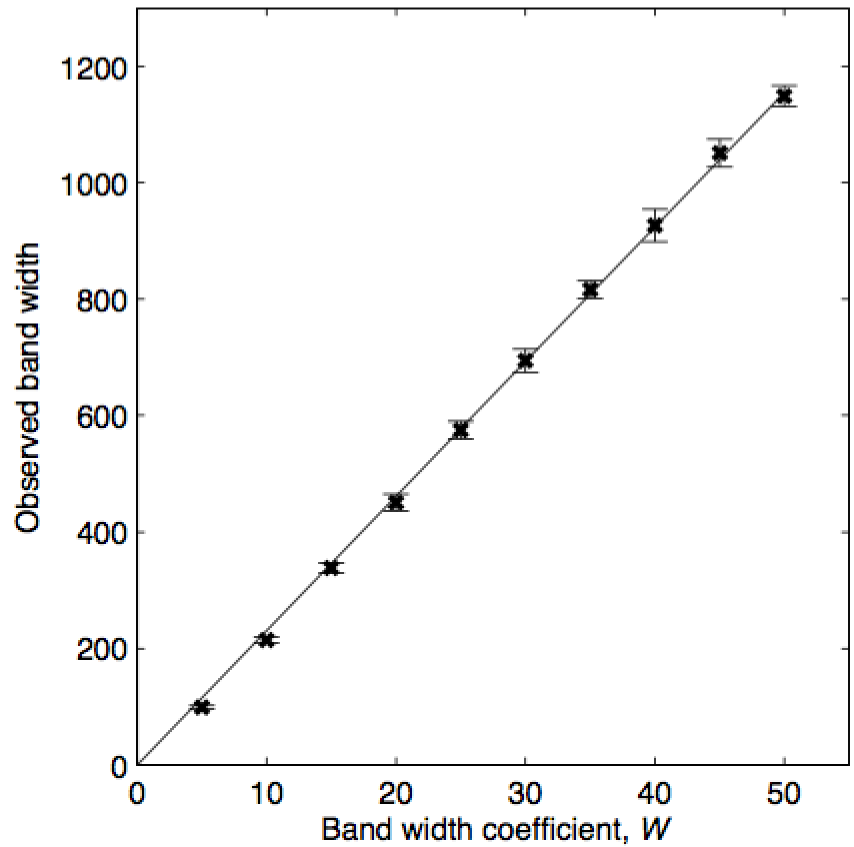
Here $\mathcal{F}$ critical size of perturbation for transition to chaos

Nonlinear effects all hidden in $\mathcal{F}$

`bandwidth coefficient' -$\mathcal{W} = \dfrac{1}{\mathrm{Re}[\lambda'_{\mathrm{max}}(V)]}$

Can compute numerically by continuation method.

Bandwidth vs. -coefficient



λ-ω system, varying ω from 1.39 to 2.77. Regression coefficient 0.9992.

End of topic 3

Main reference:
- Sherratt, Smith, R., PNAS 106: 10890-10895 (2009)

Computation and structure of spectra:
- Smith, R., Sherratt, SIAM J. Appl. Dyn. Sys. 8 (2009) 1136-1159

More on bandwidth:
- Smith, Sherratt, Phys. Rev. E 80 art. no. 046209 (2009)