A Microscopic Interpretation of the Entropy Increase Rate

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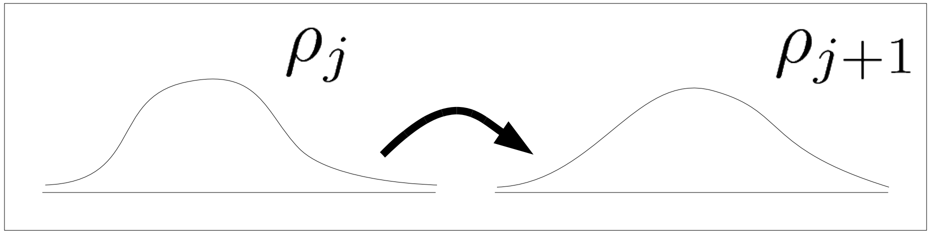
For two probability densities ρ, ρ_0 on $\mathbb{R}^d$:

$$I_h(\rho;\rho_0) := \int \rho(x)\log\rho(x)dx + \tfrac{1}{2h}d(\rho,\rho_0)^2$$

$$\begin{aligned} \rho_1 &:= \arg\min_{\rho} I_h(\rho;\rho_0) \\ \rho_2 &:= \arg\min_{\rho} I_h(\rho;\rho_1) \\ &\vdots \qquad\qquad\qquad \vdots \end{aligned}$$



Th. *(R. Jordan, D. Kinderlehrer, F. Otto 1998)*

$$\rho_{t/h} \xrightarrow{h\to 0} u \qquad \text{in } L^1(\mathbb{R}^d \times (0,\infty))$$

where u is the solution of the diffusion equation:

$$\begin{cases} \frac{\partial u}{\partial t} = \Delta u, & \mathbb{R}^d \times (0,\infty) \\ u = \rho_0, & \mathbb{R}^d \times \{t=0\} \end{cases}$$

Motivation

$$I_h(\rho;\rho_0) = \underbrace{\int \rho(x)\log\rho(x)\,dx}_{\substack{\textit{driving force:}\\ \textit{(negative) entropy}}} \quad + \quad \underbrace{\frac{1}{2h}d(\rho,\rho_0)^2}_{\substack{\textit{brake:}\\ \textit{Wasserstein distance}}}$$

How can we interpret this functional microscopically?

Look for a similar functional in ρ and ρ_0 such that the minimisers converge to u (the solution of the diffusion eq.)

ρ_0 initial probability density of particles

X_i random position of particle *i* at time *h*,
with probability density $k_h(x, x_0) = \frac{1}{(4\pi h)^{d/2}} \exp(-\frac{|x-x_0|^2}{4h})$

$L_n(x) := \frac{1}{n}\sum_{i=1}^{n} \delta_{X_i}(x)$ particle density at time *h*

Th. (*Strong Law of Large Numbers*)

$$L_n \xrightarrow{n\to\infty} u \quad a.s.$$

where u is the solution of the diffusion equation:

$$\begin{cases} \frac{\partial u}{\partial t} = \Delta u, & \mathbb{R}^d \times (0,\infty) \\ u = \rho_0, & \mathbb{R}^d \times \{t=0\} \end{cases}$$

We loose too much information in this limit!

Def. $\mathbb{P}(L_n)$ satisfies a Large Deviation Principle with rate functional J iff:

(i) $J \neq \infty$ and has compact level sets

(ii) $\frac{1}{n}\log \mathbb{P}(L_n) \xrightarrow{n\to\infty} -J$ (weak convergence by duality with continuous bounded functions)

Informally, this means that for large *n* :

$$\mathbb{P}(L_n \sim \rho) \sim e^{-nJ(\rho;\rho_0)}$$

Note that J is minimised by the most likely particle density ρ !

Relation to entropy functional

$$S(\rho) := \int \rho(x) \log \rho(x)\, dx$$

$$H(q|\mu) := \iint q(x,y) \log \frac{q(x,y)}{\mu(x,y)} dx\, dy$$

The rate functional is given by:

$$J_h(\rho;\rho_0) = \inf_{q\in\Gamma(\rho,\rho_0)} H(q|\rho_0 k_h)$$

connection for small *h*?

$$I_h(\rho;\rho_0) = S(\rho) + \tfrac{1}{2h} d(\rho,\rho_0)^2$$

Th. (*Adams, Dirr, Peletier, Zimmer; forthcoming*)

$$J_h(\cdot;\rho_0) - \tfrac{1}{4h} d(\cdot,\rho_0)^2 \xrightarrow{h\to 0} \tfrac{1}{2} S(\cdot) - \tfrac{1}{2} S(\rho_0)$$

- The Entropy-Wasserstein functional reflects the (infinitely small) probability that the system deviates from diffusion!

- Currently, my research focuses on extending this result to:
  - diffusion with drift
  - diffusion with death
  - diffusion with birth