Formal asymptotics for blowup in the Willmore flow

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Introduction

Consider an elastic surface. The elastic energy of the surface is given by

$$E = \int_{\Omega} (\alpha + \beta H^2 + \gamma K) d\mu,$$

where the integral is over the surface Ω and

- H is the *mean curvature*,
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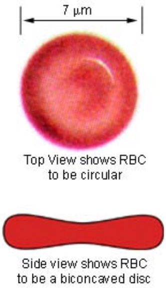


The Willmore flow I

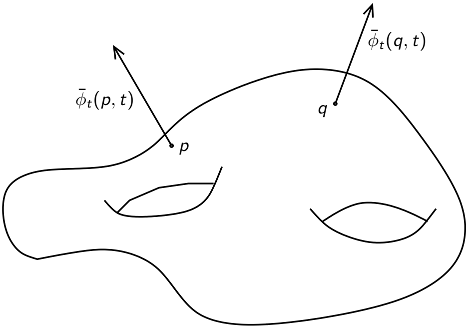
Since the integral over the Gaussian curvature is a constant (Gauss-Bonnet), minimising the bending energy comes down to minimising the Willmore functional

$$\int_\Omega H^2 d\mu.$$

Minimising this integral gives the so-called *Willmore flow*. This is a partial differential equation on a surface.

Surface evolution

Consider a moving surface $\phi(t) : M \to \mathbb{R}^3$ parametrised by t, with metric $g_{ij} = \langle \partial_i \phi, \partial_j \phi \rangle$.



Here, $\bar{\phi}_t$ is the displacement of the surface in the normal direction.

The Willmore flow II

The Willmore flow is given by

$$\bar{\phi}_t = -\Delta H - 2H(H^2 - K),$$

with

- H : mean curvature,
- K : Gaussian curvature,
- Δ : Laplace-Beltrami operator. Generalisation of Laplacian given by
$$\Delta = \frac{1}{\sqrt{\det g}} \partial_i \left(g^{ij} \sqrt{\det g} \partial_j \right).$$

Curvatures

Let κ_1 and κ_2 be the maximal and minimal curvatures of a surface at a particular point. The mean and Gaussian curvatures at that point are given by

$$H = \frac{1}{2}(\kappa_1 + \kappa_2),$$
$$K = \kappa_1 \kappa_2.$$

Willmore flow : $\bar{\phi}_t = -\Delta H - 2H(H^2 - K)$.

The sphere

Consider a sphere of radius R. If we choose the normal such that it points outwards, then

$$\kappa_1 = \kappa_2 = -\frac{1}{R},$$

everywhere. Hence,

$$H = -\frac{1}{R} \quad \text{and} \quad K = \frac{1}{R^2},$$

on the whole sphere.
We see immediately that

$$\bar{\phi}_t = -\Delta H - 2H(H^2 - K) = 0.$$

The sphere and more

- The Willmore energy for a sphere is 4π.
- The Willmore energy is scale invariant.
- The sphere is a global minimum for closed immersed surfaces.

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A surface with $\kappa_1 = -\kappa_2$, everywhere, is also a stationary solution ($H = 0$). This surface is given by the graph

$$r(z) = q \cosh\left(\frac{z}{q}\right),$$

rotated around the z-axis.

Properties of Willmore flow

- The Willmore flow is a fourth order, nonlinear, differential equation.
- Short time existence (parabolic quasi linear)
- Long time existence
  - for solutions close to a local minimum
  - for immersed spheres with Willmore energy lower equal to 8π
  - two-dimensional graphs
- If blowup occurs, the blowup profile must be stationary.

Problem

Can the Willmore flow create a singularity on a smooth surface in finite time?

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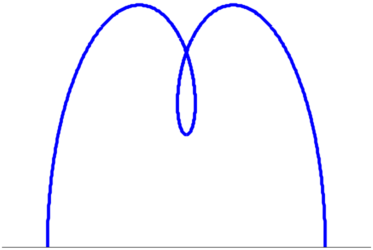
Numerical computations suggest that this can happen. Consider the following curve (a so-called Limaçon) rotated around the horizontal axes.



Numerical computations

Numerical computations suggest that a Limaçon, governed by the Willmore flow, creates a singularity in finite time.
Note

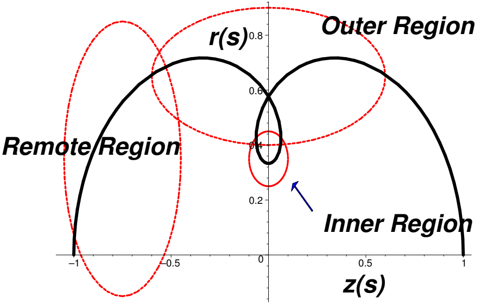
- a self intersection is not a singularity
- the tip drops with a quasi stationary rate
- the different scales

Goal of our research is to determine the rate with which the tip drops. Call this rate λ.

Different scales

There are three regions in this problem corresponding to three different scales.

- In the remote region the solution hardly moves.
- In the outer region the solution evolves (by definition) with the self similar scale $(T - t)^{1/4}$.
- In the inner region the solution is governed by the blowup rate.



Matched asymptotics

On every region we can simplify the equation differently.

- In the remote region one can use $\kappa_1 \sim \kappa_2$.
- In the outer region we use $z_r \to 0$.
- In the inner region we can use $\kappa_1 \sim -\kappa_2$.

Matching means that the solutions should behave the same in the intermediate regions.

Results

- Matching gives

$$\lambda \sim \frac{(T-t)^{1/2}}{|\ln\left(\frac{1}{T-t}\right)|^4}.$$

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- Matching gives
$$\lambda \sim \frac{(T-t)^{1/2}}{|\ln\left(\frac{1}{T-t}\right)|^4}.$$
- But we do not find a Limaçon that becomes singular.
- We find a dumbbell.

Future work

- Study the dumbbell more carefully.
  Numerically and analytically.
- Use other matching to describe the blowup of the Limaçon.
  Use moving mesh methods to study the evolution.