Stability of blowup solutions in the Ginzburg-Landau equation

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April 15, 2010

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   - Asymptotics
2. Stability
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   - Radially symmetric perturbations
   - Non-radially symmetric perturbations

The Ginzburg-Landau Equation

Ginzburg-Landau Equation

$$i\frac{\partial \Phi}{\partial t} + (1 - i\epsilon)\nabla^2\Phi + (1 + ib\epsilon)|\Phi|^{2\sigma}\Phi = 0$$

- $\Phi : \mathbb{R} \times \mathbb{R}^d \to \mathbb{C}$
- ϵ small, $\epsilon = 0$ is the Nonlinear Schrödinger Equation
- $-\frac{1}{4} < b = O(1)$
- Nonlinearity, $2 < \sigma d < 4$
- In applications only valid for small amplitude
- Interested in blowup solutions (norm solution blows up in finite time)
- $\sigma d = 2$ is critical

The Complex Ginzburg-Landau Equation

Blow-up

- Look for radially symm. solutions: $\nabla^2 = \partial_r^2 + \frac{d-1}{r}\partial_r$
- Transformation:

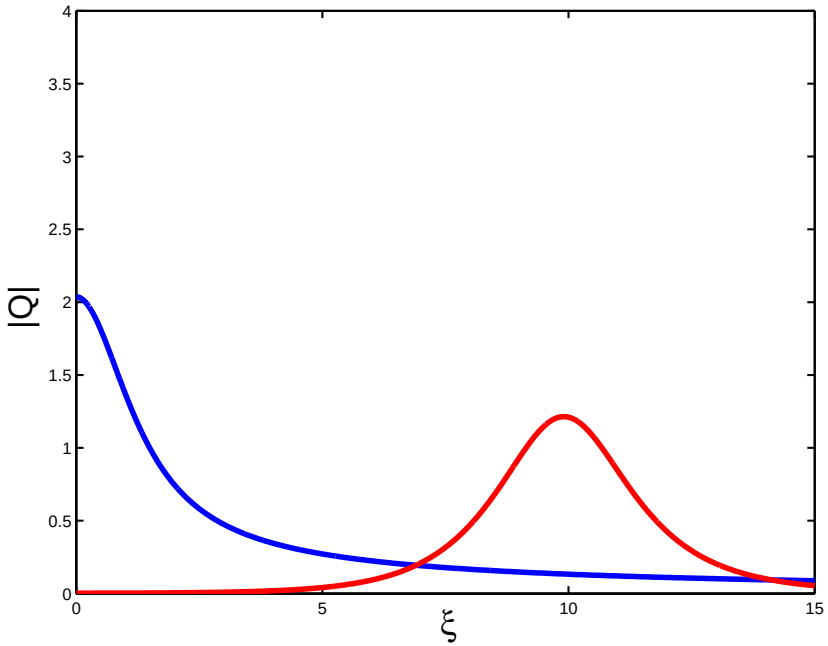
$$\Phi(x,t) := (2a(T-t))^{-\frac{1}{2}\left(\frac{1}{\sigma}+i\frac{\omega}{a}\right)} Q\Bigg(\underbrace{\frac{r}{\sqrt{2a(T-t)}}}_{=\xi \text{ new space}}, \underbrace{-\frac{1}{2a}\ln\frac{T-t}{T}}_{=\tau \text{ new time}}\Bigg)$$

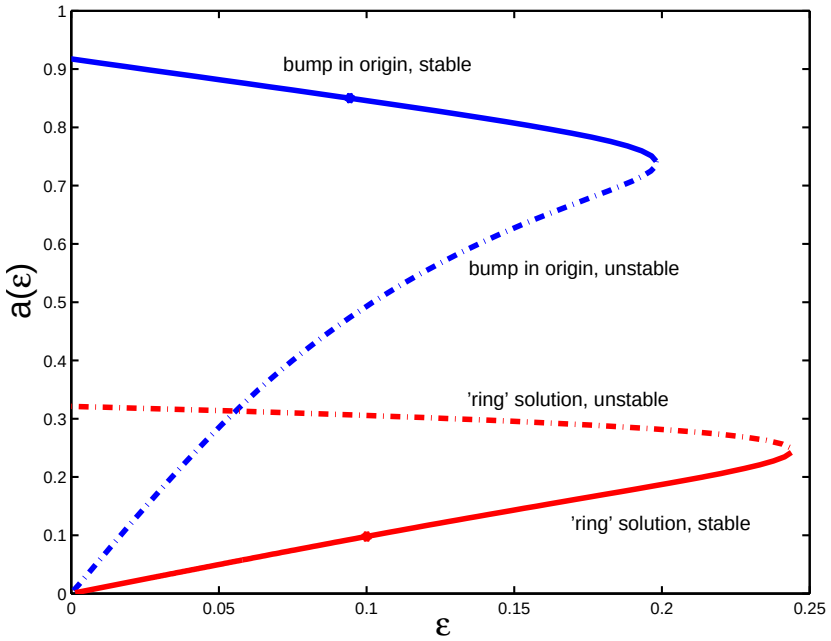
- $a > 0$

$$iQ_\tau - \omega Q + (1-i\epsilon)\left(Q_{\xi\xi} + \frac{d-1}{\xi}Q_\xi\right) + ia\left(\frac{1}{\sigma}Q + \xi Q_\xi\right)$$
$$+(1+ib\epsilon)|Q|^{2\sigma}Q = 0$$

- Look for stationary solutions
- d is a parameter (non-integer)

"Blow-up solutions" of the Rescaled Complex Ginzburg-Landau Equation

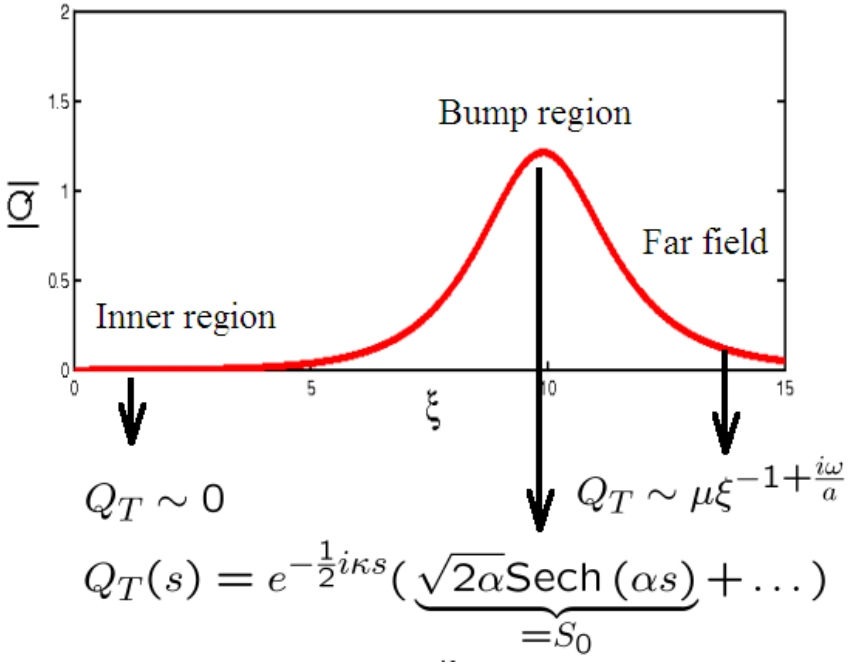




- Numerical results from Budd, Rottschäfer, Williams (2005)
- k = number of peaks, $d = 3$, $\sigma = 1$, $b = 0$
- Numerical result: ring solution is stable!
- Goal: **investigate stability with *analytic* methods**
- We look at the case $0 < a \ll 1$, $\epsilon = Ka$

Asymptotics

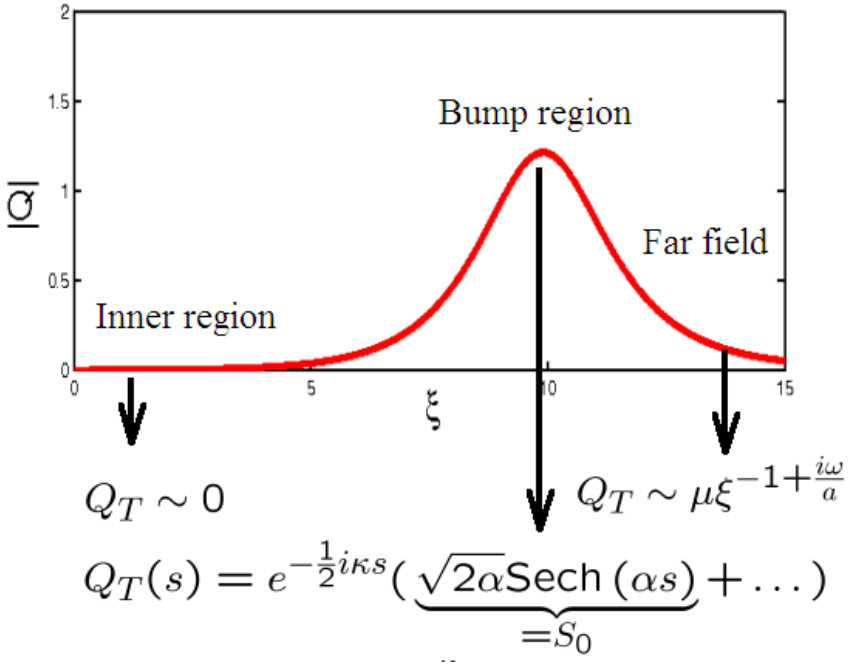
- Asymptotics ($\sigma = 1$) by Budd, Rottschäfer, Williams (2005)



- $\xi = \frac{\kappa}{a} + s$
- Important parameters: $2 < d < 4$ and $b > -\frac{1}{4}$.

Asymptotics

- Asymptotics ($\sigma = 1$) by Budd, Rottschäfer, Williams (2005)



- $\xi = \frac{\kappa}{a} + s$
- Important parameters: $2 < d < 4$ and $b > -\frac{1}{4}$.

Set-up stability w.r.t. radially symmetric perturbations

- **Standard techniques only work in the bump region**
- First concentrate on the bump region
- Add a perturbation: $\underbrace{Q(\xi)}_{\text{bump}} + \underbrace{v(\xi)e^{\lambda\tau}}_{\text{small perturbation}} \quad \rightarrow L(a)V = \lambda V$
- Stability: necessary $\mathrm{Re}\lambda \leq 0$
- Eigenvalue prob. for $v = v_1 + iv_2$ in the bump region if $a = 0$:

$$\begin{bmatrix} 0 & -\left(\partial_s^2 - \alpha + |S_0|^{2\sigma}\right) \\ \partial_s^2 - \alpha + (2\sigma+1)|S_0|^{2\sigma} & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \lambda \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}.$$

- Eigenvalues $\leftrightarrow$ localized (decaying) solutions
- Essential spectrum $\leftrightarrow$ bounded (non-decaying) solutions
- Plan: look for these solutions and match to other regions.

Eigenvalue $\lambda = 0$

- $\lambda = 0$ is an eigenvalue for $a = 0$ (no other evals)
- Problem: What happens for $a \neq 0$?
- Solution: Evans Function $E(\lambda, a)$
- Zeros of the Evans function are the eigenvalues!

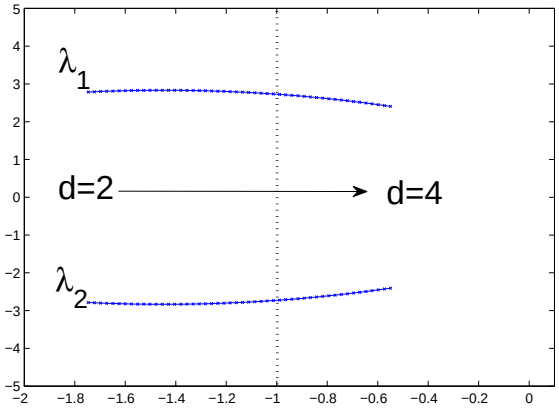
We show

$$E(\lambda, a) = \lambda^2 \left(\frac{\partial_\lambda^4 E(0,0)}{4!} \lambda^2 + \frac{\partial_a \partial_\lambda^3 E(0,0)}{1!3!} a\lambda + \frac{\partial_a^2 \partial_\lambda^2 E(0,0)}{2!2!} a^2 \right) + \dots$$

- From derivatives of Evans function we can determine $\lambda(a)$

Derivatives Evans Function

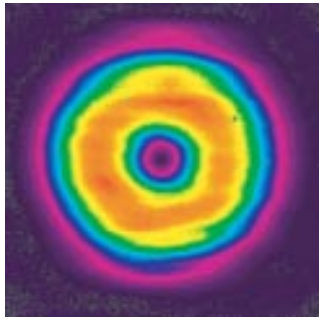
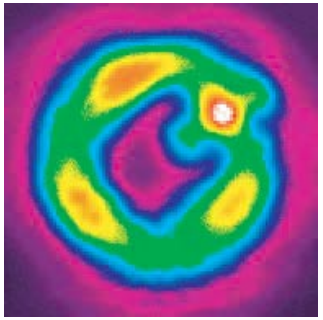
- Derivatives depend on persistence of eigenfunctions up to order a^2 and the asymptotics up to $O(a^3)$
- Working out these expressions gives hundreds of terms, thus we use Mathematica (not straightforward)
- Depends on: $d = 2 \ldots 4, b > -\frac{1}{4}$.

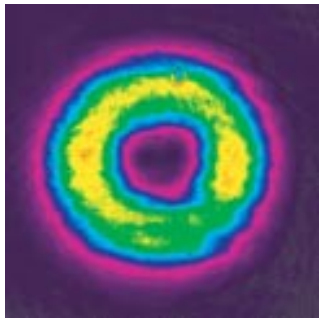
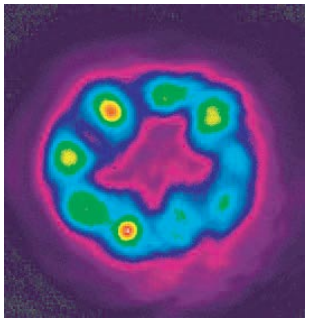


- Matching condition: $\mathrm{Re}\frac{\lambda}{a} > -1$ (algebraic decay)
- For $\mathrm{Re}\frac{\lambda}{a} = -1$: bounded solution (*on the whole domain!*)

Applications

- Laser in a nonlinear medium
- Refraction index of the medium depends on light intensity

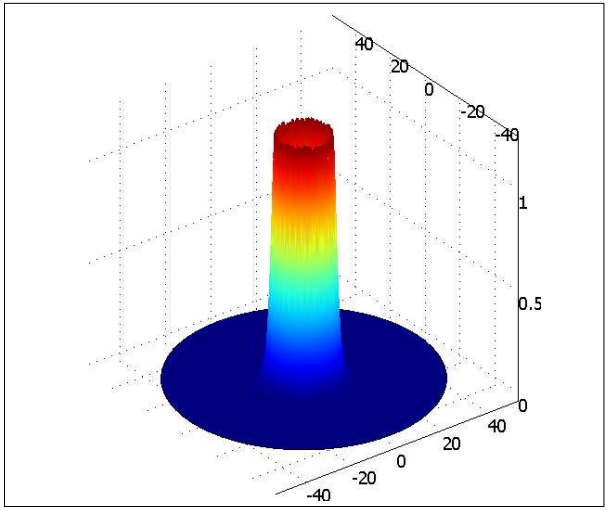
In:  , Out: 

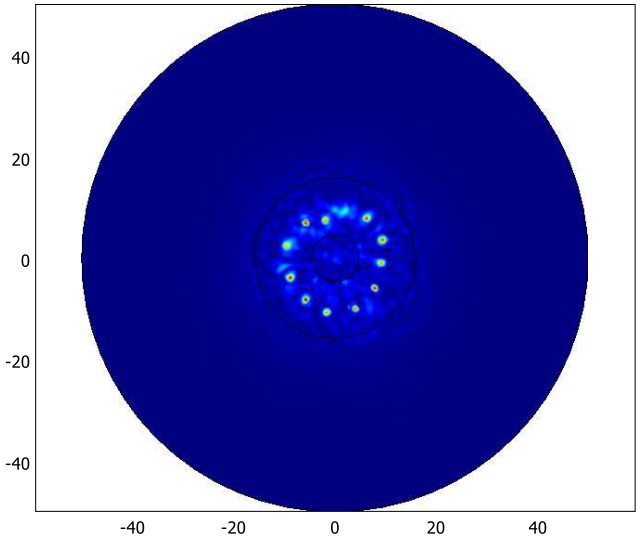
In:  , Out: 

- Figures are from Gaeta, Eliel et al 2006.

- Also allow for perturbations in the azimuthal direction

$$iQ_\tau - \omega Q + (1 - i\epsilon)\left(Q_{\xi\xi} + \frac{d-1}{\xi}Q_\xi + \frac{1}{\xi^2}Q_{\theta\theta}\right) + ia\left(\frac{1}{\sigma}Q + \xi Q_\xi\right)$$
$$+(1 + ib\epsilon)|Q|^{2\sigma}Q = 0$$





Conclusions and further research

Conclusion

- The Evans function can be used to analyse stability analytically.
- Numerical simulations indicate instability w.r.t. non-radially symmetric perturbations.
- The ring solution can break up into spots.

Questions and References

Question?

- C.J. Budd, V Rottschäfer and J.F. Williams (2005), Multi-bump, blow-up, self-similar solutions of the complex Ginzburg-Landau equation
- C.J. Budd, Asymptotics of multi-bump blow-up self-similar solutions of the nonlinear Schrödinger equation
- T. Kapitula (1999), The Evans function and generalized Melnikov integrals
- Collapse of optical vortices, Vuong, L.T. and Grow, T.D. and Ishaaya, A. and Gaeta, A.L. and t Hooft, G.W. and Eliel, E.R. and Fibich, G.