Existence of traveling waves in the Diffusive VSC model

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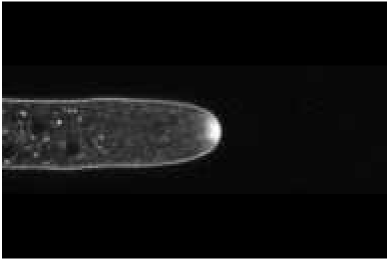
Fungal Hyphae

Fungal hyphae are tubular cells with highly localized growth in the tip of the cell. Hypha are typically $3 - 10\mu m$ wide, $50 - 400\mu m$ long, and grow at a rate of $0.1 - 6mm$ per hour.

- The VSC model was first proposed by Bartnicki-Garcia *et al.* (1989)
- Numerical work was done on the diffusive VSC model by Bela Mulder *et al.* (2006)



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[1]Movie credits: Prof. N. Read, University of Edinburgh, http://129.215.156.68/Movies/hypha.htm

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The normal velocity v_n can then be expressed in terms of the flux and the mean curvature H.

$$v_n = -\frac{F}{H}$$

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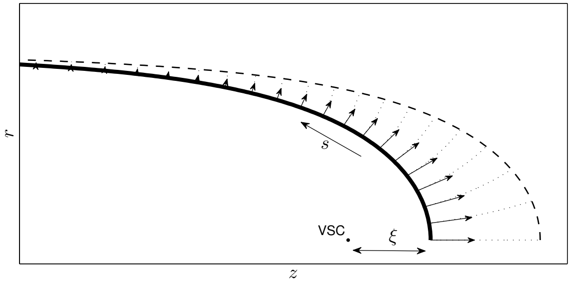
- Diffusive flux :

$$\begin{aligned} \Delta u &= -4\pi\delta(x - x_{VSC}) && \text{in } \Omega, \\ u &= 0 && \text{on } \partial\Omega. \end{aligned}$$

Traveling waves



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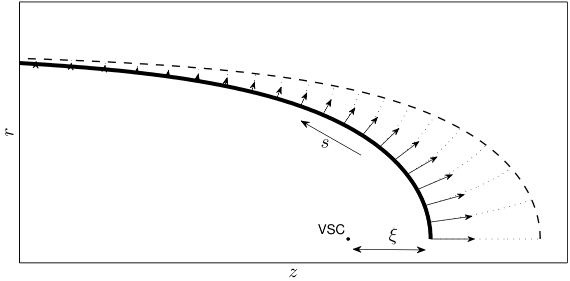

In a co-moving coordinate frame, $\partial\Omega$ is the path $(r(s), z(s))$, with s pathlength, rotated around the z axis. An equilibrium ansatz then yields

$$\frac{\mathrm{d}r}{\mathrm{d}s} = \sqrt{1 - \left(\frac{G(s)}{r(s)}\right)^2}, \qquad \frac{\mathrm{d}z}{\mathrm{d}s} = -\frac{G(s)}{r(s)},$$

where,

$$G(s) = \int_0^s F(\sigma) r(\sigma) \mathrm{d}\sigma.$$

Fixed point

To solve this we construct the following map,

$$\partial\Omega \xrightarrow{\text{PDE}} G_\xi$$

- Starting with a boundary $\partial\Omega$ we place the VSC at distance ξ from the tip and solve the Dirichlet problem to find the cumulative flux G_ξ.

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- For a specific distance ξ^* we can solve the ODE such that its solution defines a new boundary $\partial\Omega'$ with the desired properties.

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$$\begin{tikzcd} \partial\Omega \arrow[r, "\text{PDE}"] \arrow[rr, bend right, "\text{fixed point}"'] & G_\xi \arrow[r, "\text{ODE, } \xi^*"] & \partial\Omega' \end{tikzcd}$$

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A fixed point of this map is a solution to the traveling wave problem.

Schauder's fixed point Theorem

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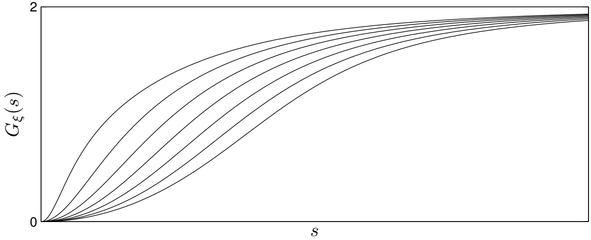
X is defined by the following restrictions on $r(s)$

$$\begin{aligned} 0 \leq r(s) &\leq 2 && \text{for all } s, \\ 0 \leq r'(s) &\leq 1 && \text{for all } s, \\ -M \leq r''(s) &\leq A && \text{for all } s, \\ r(s) &\geq s - \frac{1}{9}C^2 s^3 && \text{for } 0 \leq s \leq C^{-1}, \end{aligned}$$

for suitable constants M, A, C.

The cumulative flux



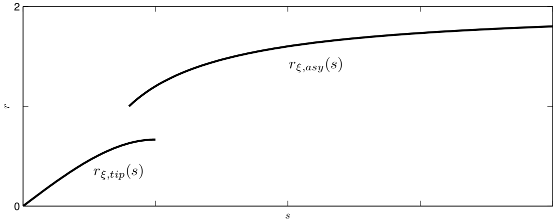
In the tip,

$$G_\xi(s) \leq 4\frac{s^2}{\xi^2}.$$

The cumulative flux is monotone in ξ.

$$\frac{\partial G_\xi}{\partial \xi}(s) < 0$$

Matching



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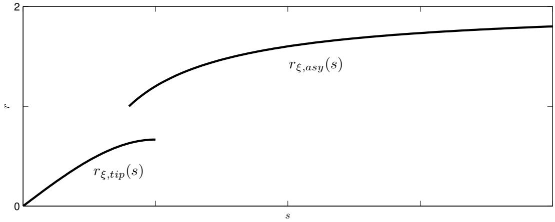

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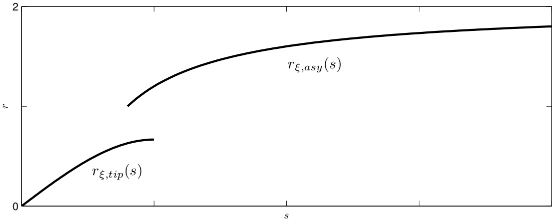

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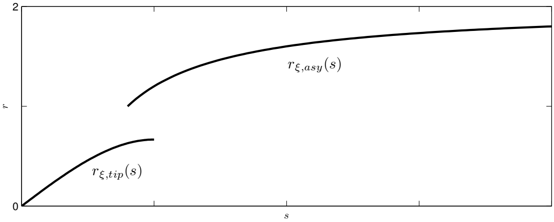

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Since these solutions are monotone in ξ, there is an unique value ξ^* such that these match.

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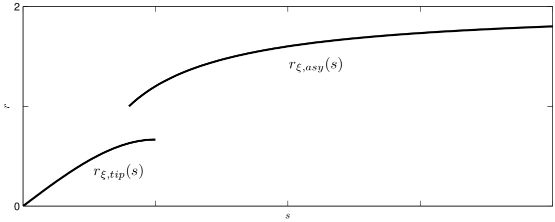

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Using the implicit function theorem, one can show that $G_\xi \to (\xi^*, r)$ is continuous.

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- Schauder's fixed point theorem does not give uniqueness of the fixed point. Does this model have a unique solution?
- Is this solution stable?
- Modifications to the model.