Non-classical travelling wave solutions to dynamic capillarity models

C.J. van Duijn[a], L.A. Peletier[b], Y. Fan[a],

P.A. Zegeling[c], H. Schuttelaars[d], I.S. Pop[a]

[a]CASA, TU/e

[b]MI, Leiden University

[c]Mathematics, Utrecht University

[d]Mathematics, TU Delft

Outline

- Motivation
- The mathematical model
- Travelling waves
- The non-linear/degenerate case
- Saturation overshoot

1. Motivation

David A. DiCarlo, Experimental measurements of saturation overshoot on infiltration, Water Resources Research, Vol. 40, W04215, doi:10.1029/2003WR002670, 2004

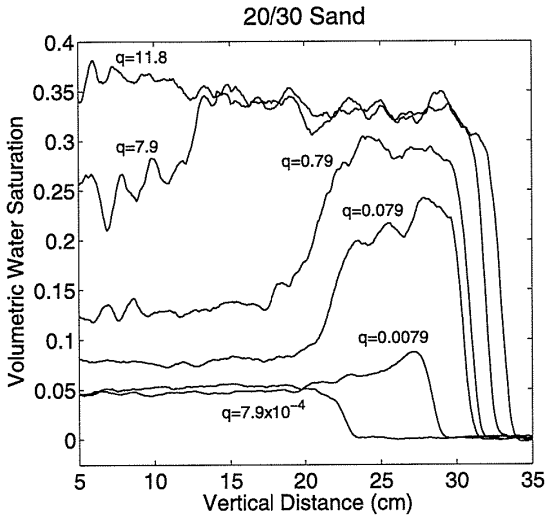


Figure 5. Snapshots of the saturation profile versus depth for six different applied fluxes in initially dry 20/30 sand (Accusand) measured using light transmission. At the highest (11.8 cm/min) and lowest (7.9×10^{-4} cm/min) fluxes the profiles are monotonic with distance and no saturation overshoot is observed, while all of the inter-

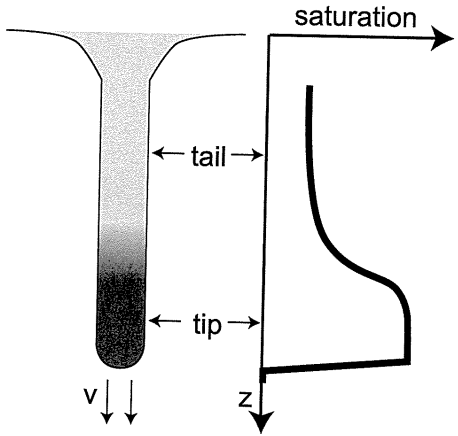
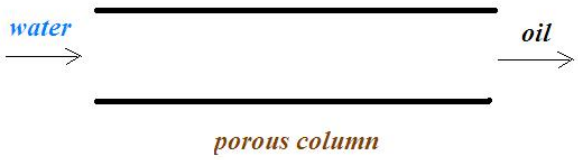


Figure 1. Cartoon of a preferential flow path and the associated saturation within the flow path. Saturation overshoot occurs when the tip saturation is greater than the tail saturation.

2. The mathematical model



Two-phase flow (wetting/non-wetting)

Homogeneous medium

Horizontal flow, one-dimensional (gravity may be included)

Equations, quantities (standard approach)

$$
\begin{aligned}
\Phi \frac{\partial S_\alpha}{\partial t} + \frac{\partial q_\alpha}{\partial x} &= 0 \quad (\alpha = w, o) \\
-q_\alpha &= K \frac{k_{r\alpha}(S_\alpha)}{\mu_\alpha} \frac{\partial p_\alpha}{\partial x} \\
S_o + S_w &= 1 \\
p_o - p_w &= p_c(S_w) \\
p_c(S_w) &= \sigma \sqrt{\frac{\Phi}{K}} J(S_w)
\end{aligned}
$$

Φ - porosity

K - absolute permeability

$S_\alpha \in [0, 1]$ - normalized saturation

q_α - specific discharge

$k_{r\alpha}$ - relative permeability

μ_α - dynamic viscosity

p_α - pressure

σ - interfacial tension

J - Leverett function

Equations, quantities (dynamic effects, Hassanizadeh & Gray)

$$
\begin{aligned}
\Phi \frac{\partial S_\alpha}{\partial t} + \frac{\partial q_\alpha}{\partial x} &= 0 \quad (\alpha = w, o) \\
-q_\alpha &= K \frac{k_{r\alpha}(S_\alpha)}{\mu_\alpha} \frac{\partial p_\alpha}{\partial x} \\
S_o + S_w &= 1 \\
p_o - p_w &= p_c(S_w) + p_c^{dyn}(S_w) \\
p_c(S_w) &= \sigma \sqrt{\frac{\Phi}{K}} J(S_w) \\
p_c^{dyn}(S_w) &= \Phi \tilde{\tau} \frac{\partial S_w}{\partial t}
\end{aligned}
$$

Φ - porosity

K - absolute permeability

$S_\alpha \in [0, 1]$ - normalized saturation

q_α - specific discharge

$k_{r\alpha}$ - relative permeability

μ_α - dynamic viscosity

p_α - pressure

σ - interfacial tension

J - Leverett function

$\tilde{\tau}$ - damping coefficient

Rem: Total velocity $q = q_w + q_o$ satisfies

$$\frac{\partial q}{\partial x} = \frac{\partial (q_w + q_o)}{\partial x} = 0$$

A$_1$: $q = q_o + q_w$ - *constant* in time (given)

A$_1$: $k_{r\alpha}$, J – monotone;

Typical choices:

$$k_{ro} = (S_o)^{1+p}; \qquad k_{rw} = (S_w)^{1+q}; \quad p, q > 0$$

$$J(S_w) = (1 - S_w)^{-\frac{1}{\lambda}}, \quad \lambda > 1$$

Scaling

Primary variable: $u = S_w$

Characteristic values: $x := \frac{x}{L},\ t := \frac{t}{T},$ with $T = \frac{\Phi L}{q}$

Balance equation: $\partial_t u + \partial_x F = 0$

$$
\begin{aligned}
F &= f(u) - N_c \lambda(u) \frac{\partial}{\partial x} \left(J(u) + N_c \tau \partial_t u \right) \\
f(u) &= \frac{k_{rw}(u)}{k_{rw}(u) + M k_{ro}(u)} \\
\lambda(u) &= k_{ro}(u) f(u)
\end{aligned}
$$

Rem: *capillary number* $N_c = \dfrac{\sigma \sqrt{K\Phi}}{\mu_o q L}$ (capillary/viscous forces)

mobility ratio $M = \mu_w / \mu_o$

$$\tau = \tilde{\tau} \frac{\mu_w q^2}{\Phi \sigma^2}$$

3. Travelling waves

$$(\mathrm{BL_{reg}})\begin{cases} \dfrac{\partial u}{\partial t} + \dfrac{\partial f(u)}{\partial x} &= \mathcal{A}_\varepsilon(u), \quad \text{in} \quad Q = \{(x,t) : x \in \mathbb{R}, t > 0\}, \\ u(\infty, t) &= u_r, \quad \text{for} \quad t > 0, \\ u(-\infty, t) &= u_\ell, \quad \text{for} \quad t > 0. \end{cases}$$

"Regularized" Riemann problem:

$$\mathcal{A}_\varepsilon(u) = \varepsilon \partial_x \big[H(u) \partial_x \big(u + \varepsilon \tau \partial_t u \big) \big].$$

Rem: $\tau = 0, \varepsilon \searrow 0$ - admissible shock solution to hyperbolic conservation laws

Q: Changing the regularization leads to different (admissible) shocks/entropy criterium?

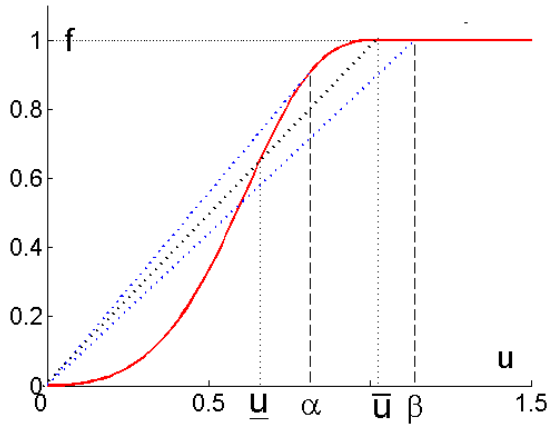
A: An admissible shock $\{u_\ell, u_r\}$ is the limit $\varepsilon \searrow 0$ of a TW solution $u = u(\eta)$ satisfying

$$(\mathrm{TW})\begin{cases} -s(u - u_r) + \{f(u) - f(u_r)\} = H(u)(u' - s\tau u''), \qquad \text{in} \qquad \mathbf{R}, \\ u(-\infty) = u_\ell, \qquad u(\infty) = u_r, \end{cases}$$

with $\eta = \dfrac{x - st}{\varepsilon}$ and $s = \dfrac{f(u_\ell) - f(u_r)}{u_\ell - u_r}$ (RH).

Q: Given $\tau > 0$, for which pairs u_ℓ, u_r and $\tau > 0$ do travelling waves exist?

Constants involved in the construction



α: "tangent point, lower bound"

β: "upper bound"

$\bar{u}$: $\alpha \leq \bar{u} < \beta$, τ - dependent

$\underline{u}$: middle intersection point

Rem:

Wave speed: $s = \frac{f(u_l)-f(u_r)}{u_l-u_r}$

Tangent point: if $u_r = 0$ we have $f'(\alpha) = \frac{f(\alpha)}{\alpha}$

Upper bound: with $g(u) := \frac{s(u-u_r)-\Big(f(u)-f(u_r)\Big)}{H(u)}$, β is s.t. $\int_{u_r}^{u_\ell} g(u)du > 0$ for all $u_\ell < \beta$.

Linear ($H(u) \equiv 1$) case

$$(\mathrm{TW}) \begin{cases} -s(u-u_r) + \{f(u) - f(u_r)\} = u' - s\tau u'', & \text{in} \quad \mathbf{R}, \\ u(-\infty) = u_\ell, \qquad u(\infty) = u_r, \end{cases}$$

Theorem 1 (Existence of TW, $u_r = 0$):

There exits $\tau_* > 0$ such that

a. If $0 \le \tau \le \tau_*$, Problem (TW) has a unique solution with $u_\ell = \alpha$ and $u_r = 0$.

b. If $\tau > \tau_*$, there exists a unique $\overline{u}_\ell(\tau) \in (\alpha, \beta)$ such that Problem (TW) has a unique solution with $u_\ell = \overline{u}_\ell(\tau)$ and $u_r = 0$.

c. $\overline{u} : [0, \infty) \to [\alpha, \beta)$ defined by

$$\overline{u}(\tau) = \begin{cases} \alpha & \text{for} \quad 0 \le \tau \le \tau_* \\ \overline{u}_\ell(\tau) & \text{for} \quad \tau > \tau_*, \end{cases}$$

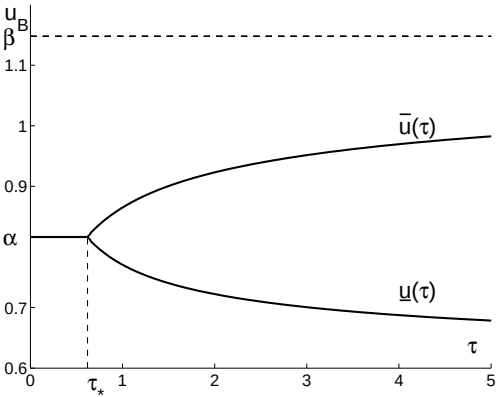
is continuous, strictly increasing for $\tau \ge \tau_*$, and $\overline{u}(\infty) = \beta$.

Rem: As $\varepsilon \searrow 0$, the (TW) becomes an admissible shock.

Case a ($0 \le \tau \le \tau_*$) provides classical shocks, dynamic effects can be neglected

Case b ($\tau \ge \tau_*$) provides non-standard shocks, violating classical entropy conditions!

The τ - $\overline{u}_\ell(\tau)$ diagram



- Numerically computed (shooting technique);

- Seek for *monotone* waves $u(\eta)$ connecting u_ℓ to $u_r = 0$;

- Then $w(u) = -u'(\eta(u))$ satisfies the *derivative equation*

$$s\tau w w' + w = su - f(u), \text{ on } (0, u_\ell),$$

with $w > 0$ on $(0, u_\ell)$, and $w(0) = w(u_\ell) = 0$.

- *Note*: first order problem, two boundary conditions. But τ and u_ℓ are related!

Rem: Computed for $M = 2$, $p = q = 1$. This gives $\alpha \approx 0.81$, $\beta \approx 1.14$. Numerically we found $\tau_* \approx 0.61$.

Rem: Diagram depends on u_r!

$$(\mathrm{TW})\begin{cases} -s(u-u_r)+\{f(u)-f(u_r)\}=u'-s\tau u'', & \text{in} \quad \mathbf{R}, \\ u(-\infty)=u_\ell, \qquad u(\infty)=u_r, \end{cases}$$

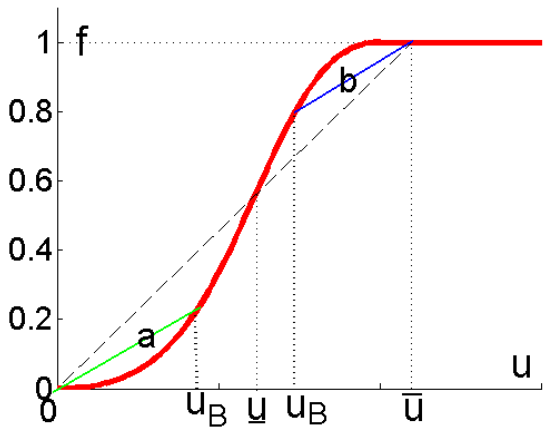
Theorem 2 ($u_r = 0$, continued):

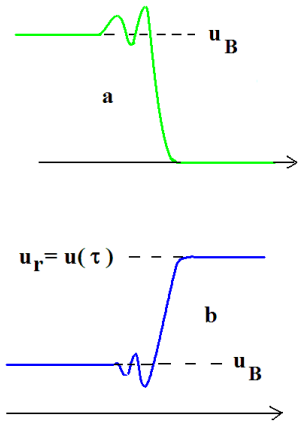
Given a $\tau > \tau_*$ we have $\overline{u}(\tau) \in (\alpha, \beta)$ and $\underline{u}(\tau) \in (0, \alpha)$. Then

a. For each $u_B \in (0, \underline{u}(\tau))$, Problem (TW) has a unique solution with $u_\ell = u_B$ and $u_r = 0$.

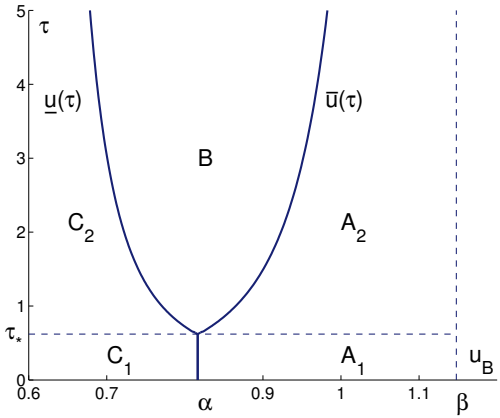
b. For each $u_B \in (\underline{u}(\tau), \overline{u}(\tau))$, Problem (TW) has a unique solution with $u_\ell = u_B$ and $u_r = \overline{u}(\tau)$; no waves are possible for $u_r = 0$.

Both waves are oscillatory.





Travelling waves connecting u_ℓ to u_r



- $(u_\ell, \tau) \in \mathcal{A}_1 \cup \mathcal{A}_2, u_r = 0$: no travelling waves.
- $(u_\ell, \tau) \in \mathcal{C}_1, u_r = 0$: existence, monotone waves.
- $(u_\ell, \tau) \in \mathcal{C}_2, u_r = 0$: existence, oscillatory waves.
- $(u_\ell, \tau) \in \mathcal{B}$:

  non-existence for $u_r = 0$;

  existence (oscillatory) for $u_\ell = \overline{u}(\tau)$, with $s(u_\ell, \overline{u}(\tau))$.

Phase plane analysis

$$s\tau u'' - u' = g(u) := \frac{s(u-u_r) - \big(f(u) - f(u_r)\big)}{H(u)} \quad \text{becomes} \quad \begin{cases} v_1' = v_2, \\ v_2' = \frac{1}{s\tau}(v_2 + g(v_1)). \end{cases}$$

Given $u_r \geq 0$ and $\tau > 0$ let $u_\ell = \bar{u}(\tau)$ and $s = \frac{f(u_\ell)-f(u_r)}{u_\ell - u_r}$. Then the system has three equilibria: $(u_r, 0)$, $(\underline{u}(\tau), 0)$, and $(\bar{u}(\tau), 0)$. The first and the last are saddle points, the intermediate is a spiral or a source.

Rem: Relation to the $\bar{u} - \tau$ diagram:

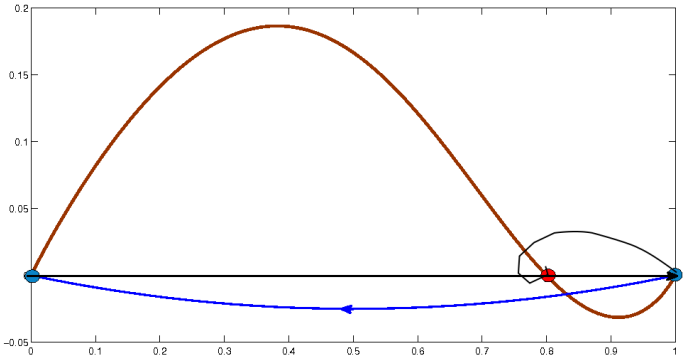
$\bar{u}(\tau)$: saddle - saddle connection, $u_\ell = \bar{u}(\tau)$ to u_r (monotone, downwards);

$\mathcal{C}_1$: source - saddle connection, $u_\ell \leq \underline{u}(\tau)$ to u_r (monotone, downwards);

$\mathcal{C}_2$: spiral - saddle connection $u_\ell \leq \underline{u}(\tau)$ to u_r (oscillatory, downwards);

$\mathcal{B}$: "superposition" of two waves, a spiral to saddle connection $u_\ell \in (\underline{u}(\tau), \bar{u}(\tau))$ to $\bar{u}(\tau)$ (oscillatory, upwards), and a saddle - saddle connection, $u_\ell = \bar{u}(\tau)$ to u_r (monotone, downwards).

Phase plane associated to the travelling wave: saddle to saddle connection (blue), spiral to saddle connection (brown)



Numerical experiments

$$\begin{cases} \dfrac{\partial u}{\partial t} + \dfrac{\partial f(u)}{\partial x} = \varepsilon \dfrac{\partial^2 u}{\partial x^2} + \varepsilon^2 \tau \dfrac{\partial^3 u}{\partial x^2 \partial t} & \text{in} \quad \mathbf{R} \times \mathbf{R}^+, \\ u(x,0) = u_B \tilde{H}(-x) & \text{for} \quad x \in \mathbf{R}, \end{cases}$$

with $\tilde{H}$ - smooth approximation of the Heaviside graph.

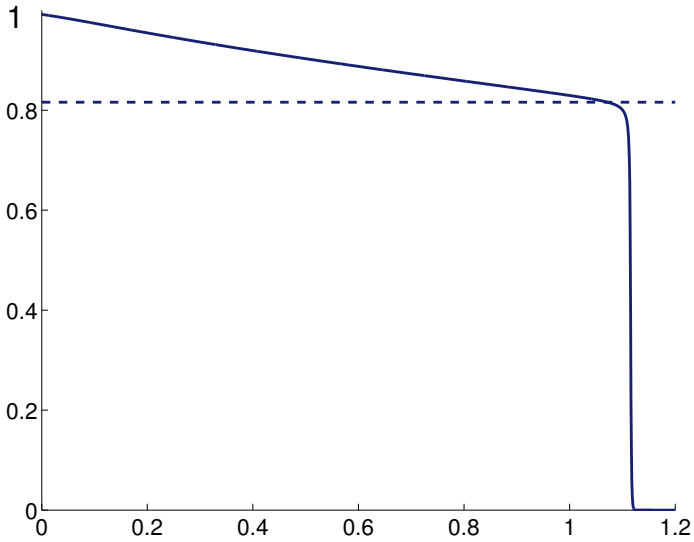
Numerical scheme:

Implicit for higher order terms, first order in time & finite differences;

Explicit for convection, *minmod* flux limiting scheme, upwind & Richtmyer.

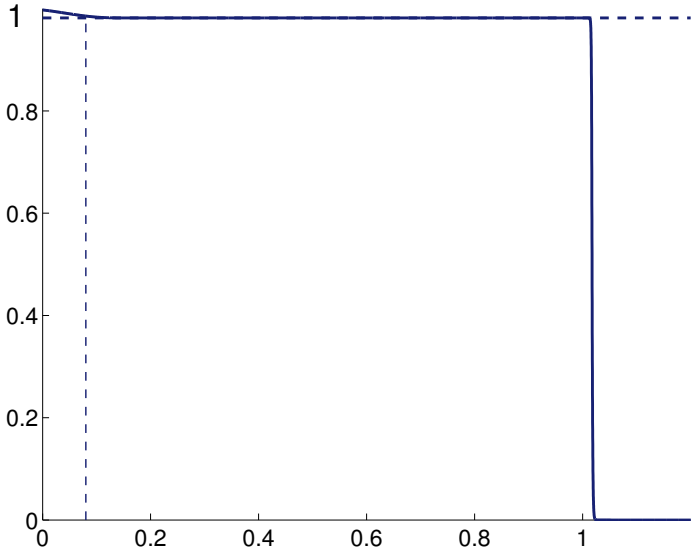
Examples:

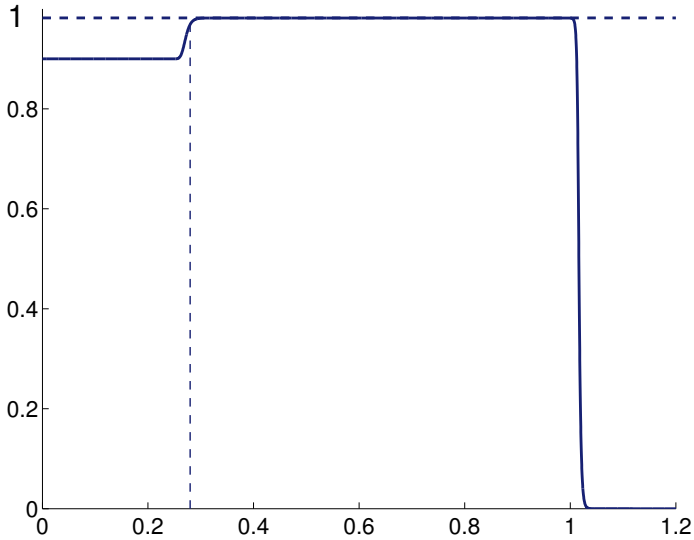
Case $\mathcal{A}_1$: $\tau = 0.2$, $u_B = 1.0$



Rem: Since $\tau < \tau_*$, the solution first decays to $\alpha = \bar{u}(\tau)$.

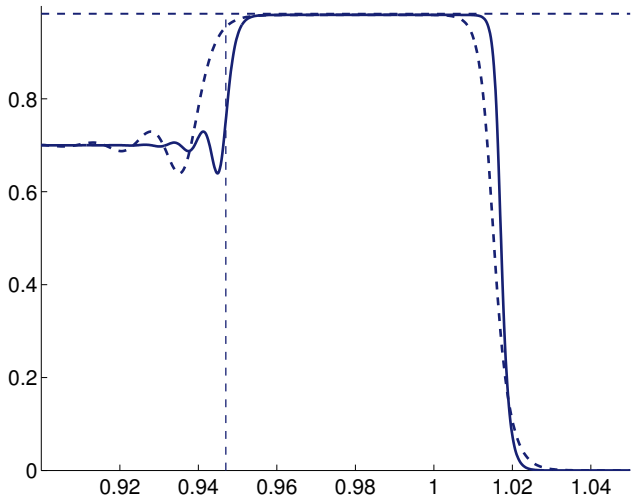
Non-standard: $\tau = 5 > \tau_*$ cases $\mathcal{A}_2$, $\mathcal{B}$:

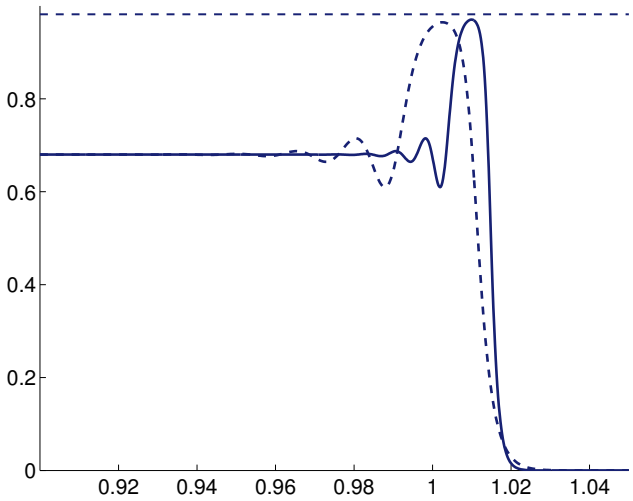




Rem: Plateau value ($\overline{u} \approx 0.98$) agrees excellently with the diagram: $u(\tau = 5) \approx 0.98$!

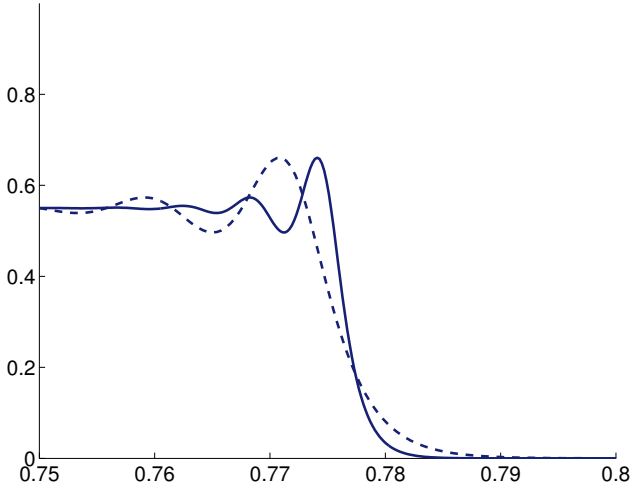
Non-standard: $\tau = 5 > \tau_*$ cases $\mathcal{B}, \mathcal{C}_2$:





Rem: As $u_B \searrow \underline{u}(\tau = 5) \approx 0.68$ the plateau vanishes and the solution transforms into an (oscillatory) front $\{u_B, 0\}$!

"Nearly"-standard: $\tau = 5 > \tau_*$ case $\mathcal{C}_2$:



4. The non-linear/degenerate case

$$(\mathrm{BL}_{\mathrm{reg}})\begin{cases} \dfrac{\partial u}{\partial t} + \dfrac{\partial f(u)}{\partial x} &= \mathcal{A}_\varepsilon(u), \quad \text{in} \quad Q = \{(x,t) : x > 0, t > 0\}, \\ u(x,0) &= 0, \quad \text{for} \quad x > 0, \\ u(0,t) &= u_B, \quad \text{for} \quad t > 0, \end{cases}$$

Regularization ($J(u) = u$):

$$\mathcal{A}_\varepsilon(u) = \varepsilon \partial_x \big[H(u) \partial_x \big(u + \varepsilon \tau \partial_t u \big) \big]$$

with

$$f(u) = \frac{k_{ro}(u)}{k_{ro}(u) + M k_{rw}(u)} \quad \text{and} \quad H(u) = k_{rw}(u) f(u),$$

for $k_{ro}(u) = u^{1+p}$ and $k_{rw}(u) = (1-u)^{1+q}$.

Rem:

Linear case: $H(u) = 1$ everywhere.

Non-linear case: H is not constant;

Degeneracy ($H(u) = 0$) occurs if $u = 0$ (fully oil saturated, no water) or $u = 1$ (fully water saturated).

Physically relevant regime: $0 \leq u \leq 1$; $H(u \leq 0) = H(u \geq 1) = 0$.

TW approach: $u = u(\eta)$ with $\eta = \frac{x-st}{\varepsilon}$ satisfies (after integration over (η, ∞))

$$(\mathrm{TW}) \begin{cases} -s(u-u_r) + \{f(u) - f(u_r)\} = H(u)(u' - s\tau u''), & \text{in} \quad \mathbf{R}, \\ u(-\infty) = u_\ell, \qquad u(\infty) = u_r, \end{cases}$$

with $s = \frac{f(u_\ell)-f(u_r)}{u_\ell - u_r}$ (RH).

Rem: As in the linear case,

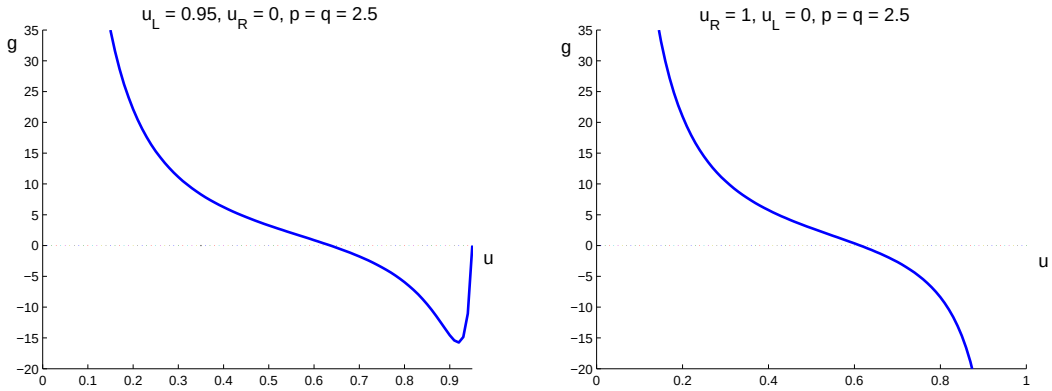
$$s\tau u'' - u' = g(u) := \frac{s(u-u_r) - (f(u) - f(u_r))}{H(u)}.$$

For monotone waves, $w(u) = -u'(\eta(u))$ satisfies

$$\text{(ODE)} \quad s\tau w w' + w = g(u), \text{ on } (u_r, u_\ell),$$

with $w > 0$ on (u_r, u_ℓ), $w(u_r) = 0$ and (if possible) $w(u_\ell) = 0$.

Rem: Blow up for g as $u \longrightarrow 0$ or $u \longrightarrow 1$!



The graph of g, $\alpha < u_l < 1$ (left) and $u_l = 1$ (right).

Results ($u_\ell > u_r$)

Depend on p, q, u_ℓ and u_r:

- Existence for $p \in (0,1)$, or $u_r > 0$; $q \in (0,1)$, or $u_\ell < 1$;
- Solution remains bounded $0 \leq u \leq 1$ ("physically relevant");
- Non-existence for $p \geq 1$ and $u_r = 0$, respectively $q \geq 1$ and $u_\ell = 1$;
- Existence of weak solutions & essential bounds: Mikelić (2010)

Note: Necessary condition for existence: $s\tau u'' - u' = g(u)$ admits a solution if

$$0 < \int_{u_r}^{u_\ell} g(u)du < \infty, \quad \text{where} \quad g(u) := \frac{s(u-u_r) - (f(u) - f(u_r))}{H(u)}.$$

Consider $u_\ell \geq \alpha$ and define

$$\mathcal{G} : [\alpha, 1] \to 1, \qquad \mathcal{G}(u_l) := \int_{u_r}^{u_\ell} g(u)du.$$

Further

$$\beta = \max\{u_l \in [\alpha, 1] / \mathcal{G}(u_l) \geq 0\}.$$

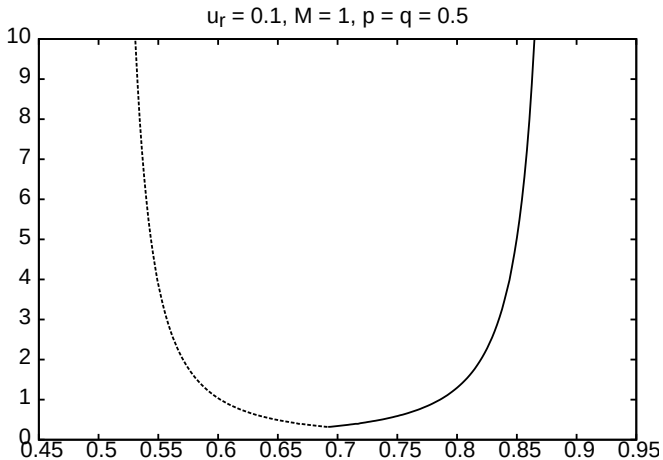
Rem: No travelling waves if $u_\ell > \beta$!

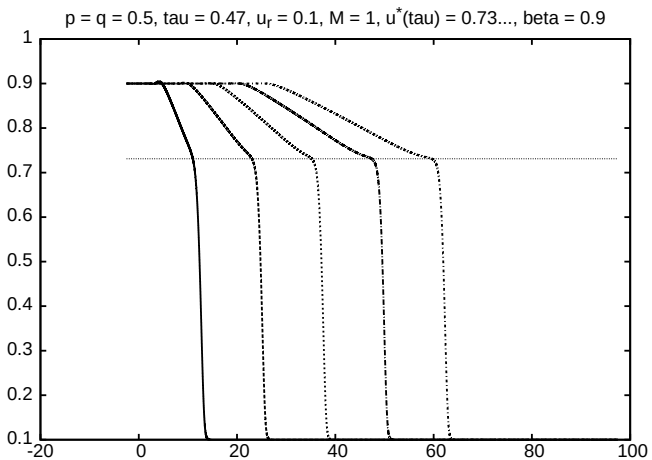
Non-linear, non-degenerate case: $u_r > 0$, $\beta < 1$

Similar to the linear case, we have:

- Existence of (monotone) waves with $u_\ell \in [u_r, \alpha]$ if $\tau < \tau_*$ (small dynamic effects);
- For any $\tau > \tau_*$ a unique $u_\ell = \bar{u}(\tau) \in (\alpha, \beta)$ exists for which (monotone) travelling waves are possible;
- If $\tau > \tau_*$, (oscillatory) travelling waves also exist for $u_\ell \in (u_r, \underline{u}(\tau))$.

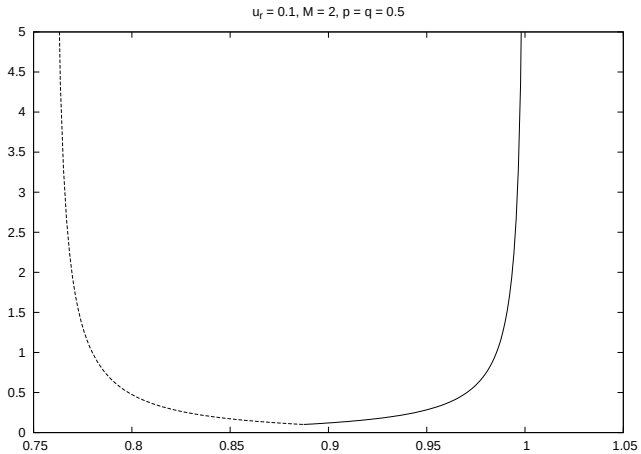
The case $\mathcal{A}_2$

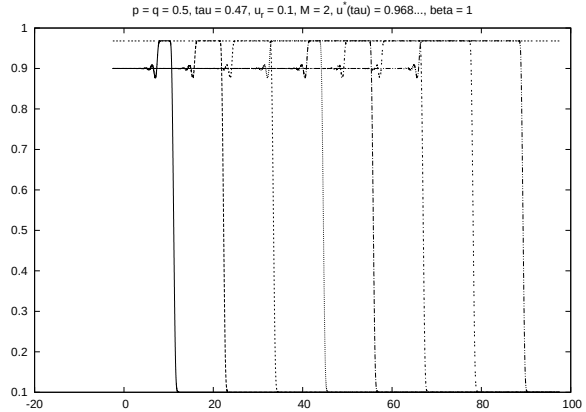




Rem: Here $\alpha < 0.7$, whereas $\tau = 0.47 > \tau_*$ giving $\bar{u}(\tau) \approx 0.73$. Since $u_B > \bar{u}(\tau)$ (case $\mathcal{A}_2$) the solution approaches first $\bar{u}(\tau)$ before becoming a front $\bar{u}(\tau) \searrow u_r = 0$. For $\tau \leq \tau_*$ the front is $\alpha \searrow u_r$, but $\alpha < 0.7$.

The case $\mathcal{B}$





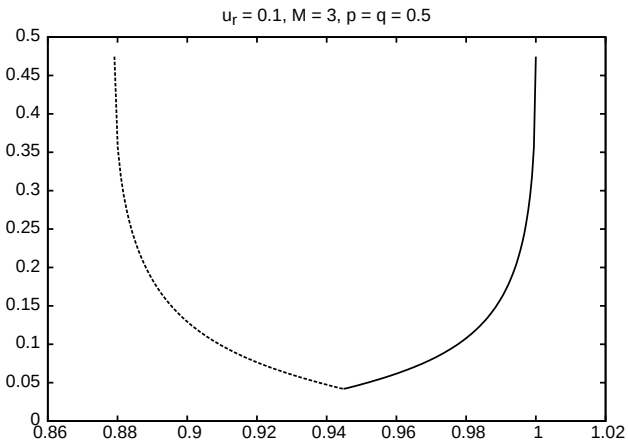
Rem: Here $\tau = 0.47 > \tau_*$ giving $\bar{u}(\tau) \approx 0.97 > \alpha$. With $u_B < \bar{u}(\tau)$, two waves are observed $u_B \nearrow \bar{u}(\tau)$ (oscillatory) and $\bar{u}(\tau) \searrow u_r$ (monotone).

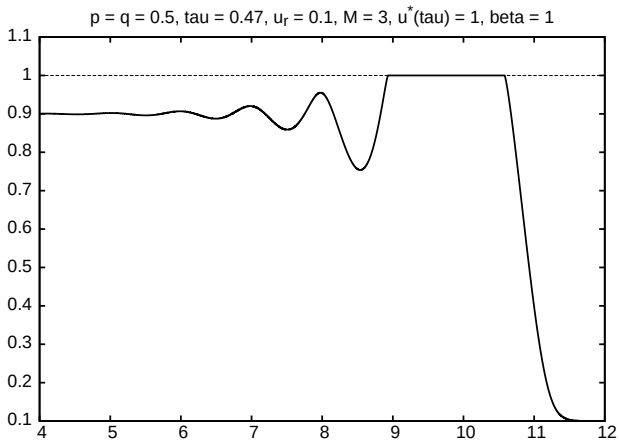
Degenerate case: $u_r > 0$, $\beta = 1$

If $\beta = 1$, then $\bar{u}(\tau) \nearrow 1$ as τ is increasing.

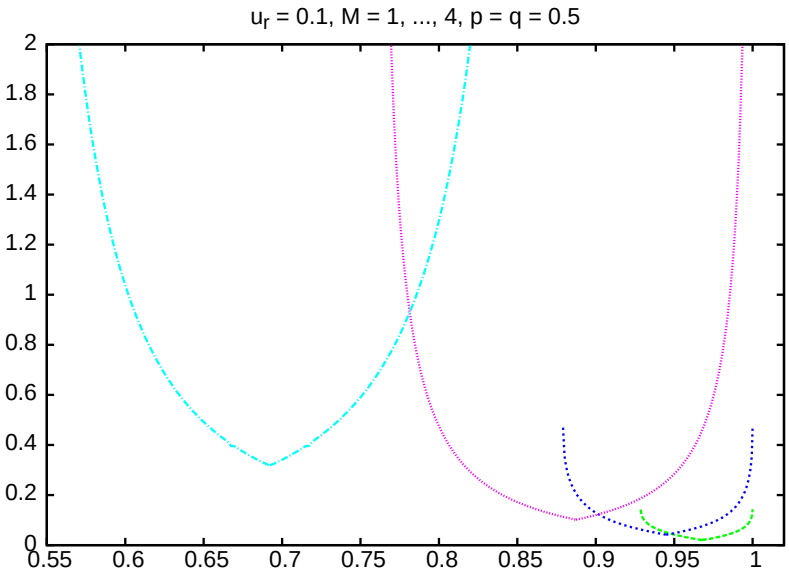
Theorem (upper bound for τ):
If $\beta = 1$, there exist a $\tau^* > \tau_*$ (both depending on u_r) s.t. for any $\tau \in (\tau_*, \tau^*)$ a unique $u_\ell = \bar{u}(\tau) \in (\alpha, 1)$ exists allowing for monotone travelling waves $\{u_\ell, u_r\}$.

Rem: At τ^*, $u_\ell = 1$ (degeneracy, $H(1) = 0$).





Rem: Here $\tau^* \approx 0.47$, thus $\bar{u}(\tau) \approx 1$.



Diagrams $\tau - u_\ell = \bar{u}(\tau)$, for $M = 1, 2, 3, 4$, $u_r = 0.1$, $p = q = 0.5$:

$M = 1, 2$, when $\beta < 1$.
For any $\tau > 0$ a unique $\bar{u}(\tau) < 1$ exists allowing for smooth, monotone TW $\{u_\ell = \bar{u}(\tau), u_r\}$.

$M = 3, 4$, when $\beta = 1$.
Smooth waves are only possible if $\tau < \tau^*$ (here $\tau^* \approx 0.47$, respectively $\tau^* \approx 0.14$).

Q: What if $\tau \geq \tau^*$?

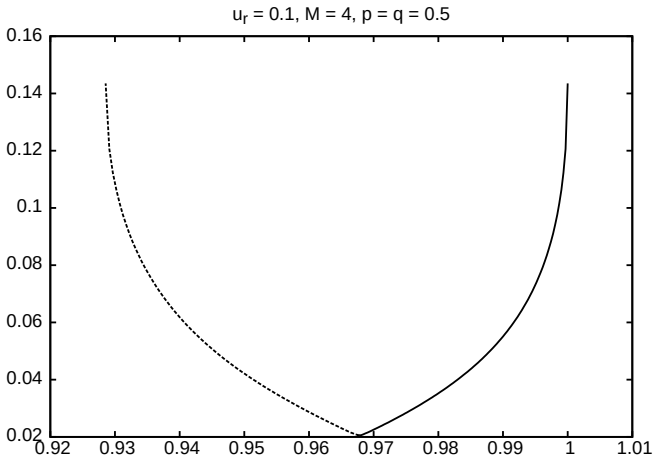
A: "Sharp waves": For some $\eta_1 \in \mathbb{R}$ we have $u \in C(\mathbb{R}) \cap C^1(\mathbb{R}\backslash\{\eta_1\})$, and

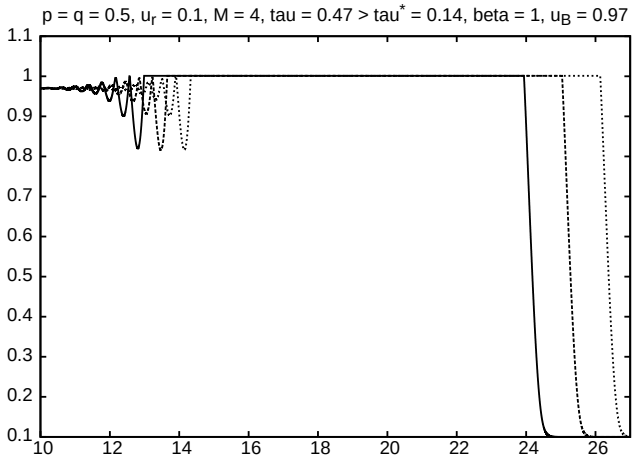
$$u(\eta \leq \eta_1) = 1, \quad \text{while} \quad u(\eta > \eta_1) \in (u_r, 1),$$

leading to (non-standard) waves $\{1, u_r\}$.

Rem: Kink at η_1, where $u = 1$. There $u'(\eta_1 - 0) = 0 > u'(\eta_1 + 0)$. Further, at η_1 the capillary pressure $p = u + \tau u'$ becomes discontinuous!

Degenerate: $u_r = 0.1$, but $\tau = 0.47 > \tau^* \approx 0.14$ yielding $u_\ell = 1$:





Rem: For monotone waves, $w(u) = -u'(\eta(u))$ satisfies

$$\text{(ODE)} \quad s\tau w w' + w = g(u) := \frac{s(u-u_r) - (f(u) - f(u_r))}{H(u)}, \text{ on } (u_r, 1),$$

with $w > 0$ on $(u_r, 1)$, and $w(u_r) = 0$. However, $w(1) > 0$!

Here we have $w(1) \approx -u'(\eta_1) \approx -1.64$!

Degenerate case, $u_r = 0$

If $\beta < 1$, then (as in the non-degenerate case):

- Standard (monotone) waves $\{u_\ell, 0\}$, if $u_\ell \in (0, \alpha)$ and $\tau < \tau_*$ (small dynamic effects);
- For any $\tau > \tau_*$ a unique $u_\ell = \bar{u}(\tau) \in (\alpha, \beta)$ allowing for:
  - monotone waves connecting u_ℓ to 0;
  - oscillatory waves connecting $u_\ell \in (0, \underline{u}(\tau))$ to 0.

Rem: These waves are smooth (no kinks)!

If $\beta = 1$, then there exists a $\tau^* = \tau^*(0) > \tau_*$ s.t.

- For any $\tau \in (\tau_*, \tau^*)$, a unique $u_\ell = \bar{u}(\tau) \in (\alpha, 1)$ exists allowing for travelling waves connecting $u_\ell = \bar{u}(\tau)$ to 0. Such waves are smooth and monotone!
- If $\tau \geq \tau^*$, then only *sharp* waves are possible.

Note: Two degeneracy points, $u_r = 0$ and $u_\ell = 1$, many non-smooth solutions are possible (one kink for each degeneracy; there are many slopes possible for each of the kinks).

Q: How to select a wave?

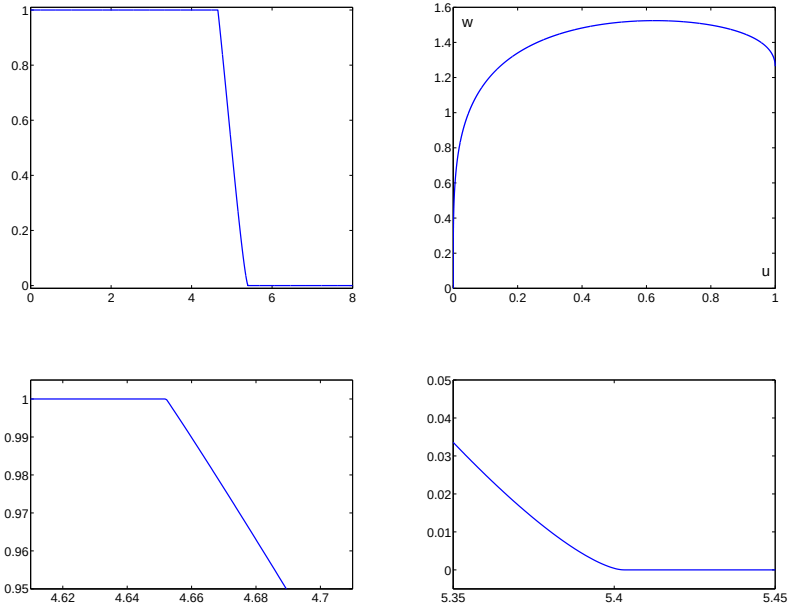
A: As limit $\delta \searrow 0$ of waves $\{1, \delta\}$!

This leads to waves satisfying for some $\eta_1 < \eta_0$, $u \in C(\mathbb{R}) \cap C^1(\mathbb{R} \backslash \{\eta_1\})$,

$$u(\eta \leq \eta_1) = 1, \quad \text{while} \quad u(\eta_1 < \eta < \eta_0) \in (0, 1), \quad \text{and} \quad u(\eta \geq \eta_0) = 0.$$

Rem: Kink at η_1, where $u = 1$: $u'(\eta_1 - 0) = 0 > u'(\eta_1 + 0)$. However, the wave is smooth at η_0; it decays faster than quadratically to 0.

Double degeneracy: $u_r = 0$, $u_\ell = 1$



Rem: Kink ($u' = -1.27$) at η_1, smooth transition ($u' = 0$) at η_0; here $w(1) = 1.26$.

5. Saturation overshoot

Infiltration problem: $u(t,0) = u_B$, $u(t,"\infty") = u_r$

- The standard Richards model provides *monotone* profiles:

$$\partial_t u + \nabla \cdot \big(K(u)\mathbf{g}\big) = \nabla \cdot \big(K(u)\nabla P_c(u)\big)$$

- Higher order terms: "dynamic capillarity" C. Cuesta, J. Hulshof, C.J. van Duijn (2000), A. Egorov, J. Nieber, R. Dautov (2003)

$$\partial_t u + \nabla \cdot \big(K(u)\mathbf{g}\big) = \nabla \cdot \big(K(u)\nabla(P_c(u) + \tau\partial_t u)\big)$$

  "phase field" L. Cueto-Felgueroso, R. Juanes (2009)

$$\partial_t u + \nabla \cdot \big(K(u)\mathbf{g}\big) = \nabla \cdot \big(K(u)\nabla(P_c(u) + \Delta u)\big)$$

- Multi-phase (percolating/non-percolating) systems R. Hilfer, F. Doster, P. Zegeling (2009)

Rem: The convective term K is *convex*!

One phase flow: water, K is convex!

$$\begin{cases} v_1' = v_2, \\ v_2' = \frac{1}{s\tau}(v_2 + g(v_1)). \end{cases}$$

- Only two equilibria are allowed, regardless extension;
- With $u_1 < u_2 < u_3$, upwards wave $\{u_2 \nearrow u_3\}$ travel faster than downwards wave $\{u_3 \searrow u_1\}$

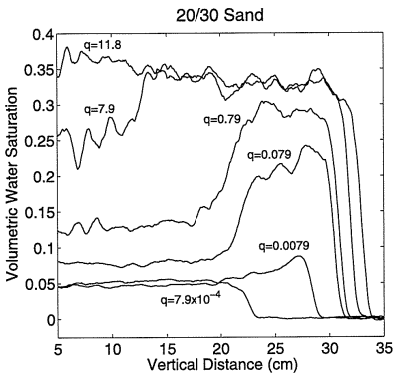
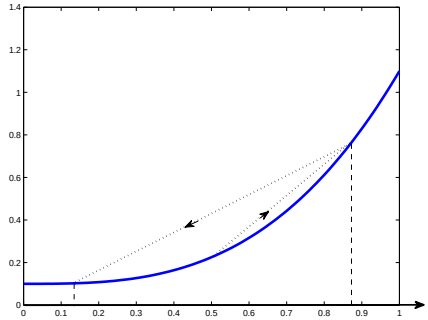
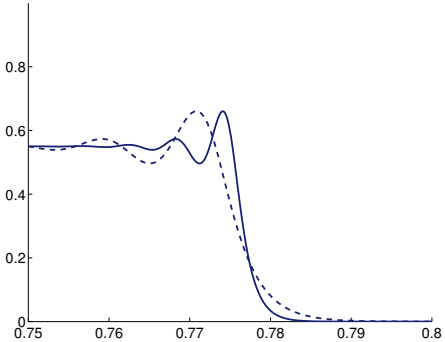


Figure 5. Snapshots of the saturation profile versus depth for six different applied fluxes in initially dry 20/30 sand (Accusand) measured using light transmission. At the highest (11.8 cm/min) and lowest (7.9×10^{-4} cm/min) fluxes the profiles are monotonic with distance and no saturation overshoot is observed, while all of the inter-

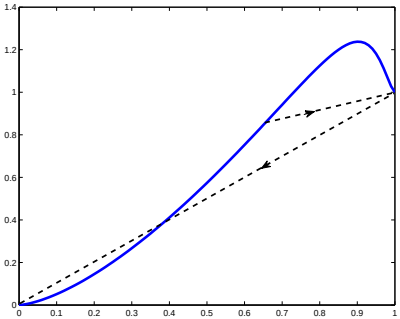


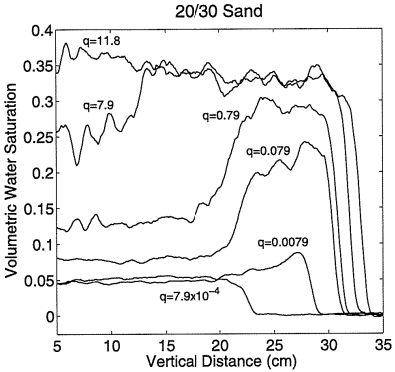


Two-phase flow: water and air convex-concave flux, $f(u)(V_t + k_a(u)\mathbf{g})$!

$$\begin{cases} v_1' = v_2, \\ v_2' = \frac{1}{s\tau}(v_2 + g(v_1)). \end{cases}$$

- More than two equilibria are allowed;
- With $u_1 < u_2 < u_3$, upwards wave $\{u_2 \nearrow u_3\}$ can be combined with downwards wave $\{u_3 \searrow u_1\}$





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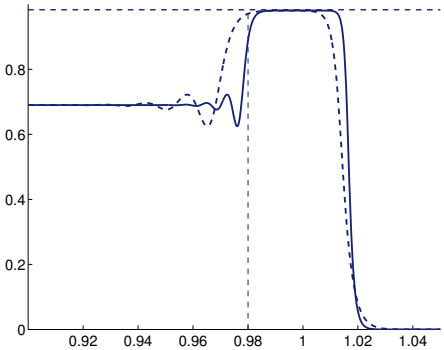

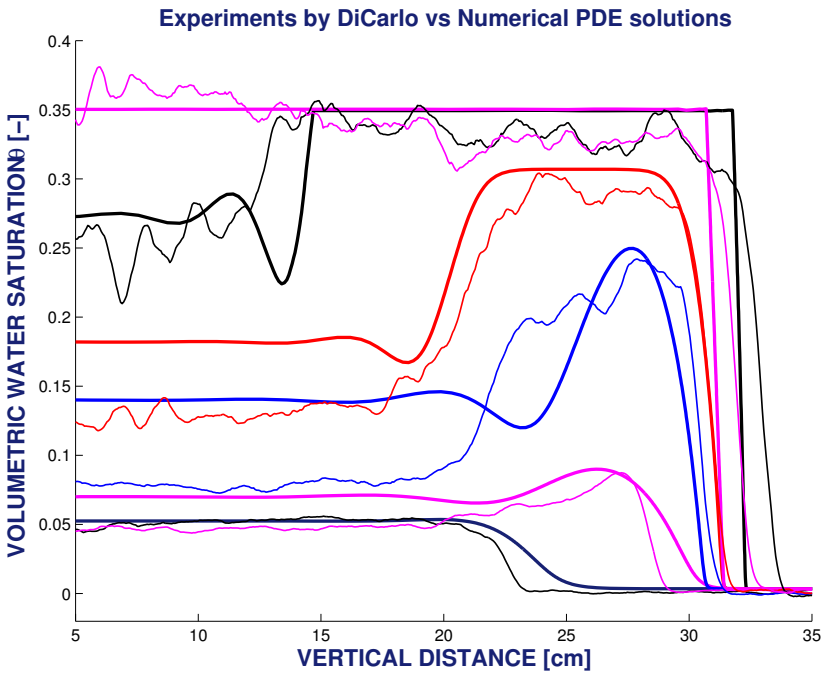

Rem:

Brooks-Corey (two phase), parameters for 20/30 sand, total velocities as in the experiments,

Fully nonlinear, degenerate model, τ is fitted (agrees with values reported before).

Conclusions and perspectives

- Travelling waves for different regularizations (equilibrium/dynamic capillary pressure) of the Buckley-Leverett model
- Nonstandard entropy solutions to the hyperbolic Buckley-Leverett equation
- Explains the occurrence of experimental results, ruled out by equilibrium models
- Degenerate terms are required for remaining inside the physically relevant regime
- Parametrization is essential (what if $p \geq 1$ or $q \geq 1$)?
- Agreement with experimental work (saturation overshoot)?
- Mathematical analysis (existence/uniqueness?) of weak solutions
- Numerical analysis (appropriate/convergent numerical schemes)
- Non-smooth data (jumps)
- Heterogeneous media
- Upscaling (pore to core)

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