Busse balloons

from the reversible Gray-Scott model to nonreversible Klausmeier models with nonlinear diffusion

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in cooperation with
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Direct motivation: vegetation patterns. Below in Sahel

Figure: Spotted patterns

The Model

- We model these patterns by

$$
\begin{aligned}
U_t &= D U^{\gamma}{}_{xx} + A(1-U) - UV^2 + C U_x \\
V_t &= \delta^2 V_{xx} - BV + UV^2 .
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- It also generalises the Gray-Scott system ($C = 0, D = 1, \gamma = 1$)

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Birth of patterns

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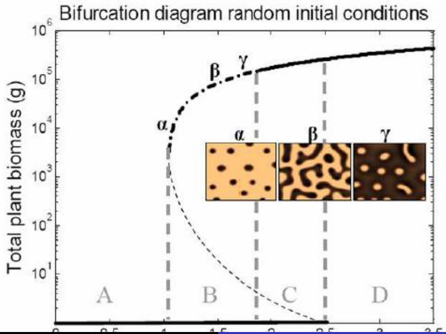
- We consider A as the critical parameter. A corresponds to the rainfall.
- For A below some critical value A_c, the (stable) homogeneous background state looses its stability with respect to periodic perturbations. This is called a Turing-Hopf bifurcation.



The Eckhaus band: symmetric (Gray-Scott) case

Let A be the amplitude of the emerging pattern. Then the pattern can be written out as

$$U \propto U_0 + \varepsilon A(\xi,\tau)e^{ik_c x} + \text{c.c.} + \text{h.o.t.}$$

In the case of the Gray-Scott system, e.g., we find for A a real GLE:

$$A_\tau = \frac{2}{\sqrt{b}}A + 2\sqrt{2}A_{\xi\xi} - \frac{2}{9}(10\sqrt{2}-7)|A|^2 A$$

A band of spatially periodic patterns bifurcates for

$$k \in (k_c - \varepsilon\sqrt{r}, k_c + \varepsilon\sqrt{r})$$

Stable patterns only exist in the Eckhaus band

$$k \in (k_c - \varepsilon\sqrt{\frac{r}{3}}, k_c + \varepsilon\sqrt{\frac{r}{3}})$$

Let k be the wavenumber of a pattern born at the Turing bifurcation. For $|A - A_c| = \mathcal{O}(\varepsilon)$, the Eckhaus band describes a region in (A, k)-space where stable patterns exist.

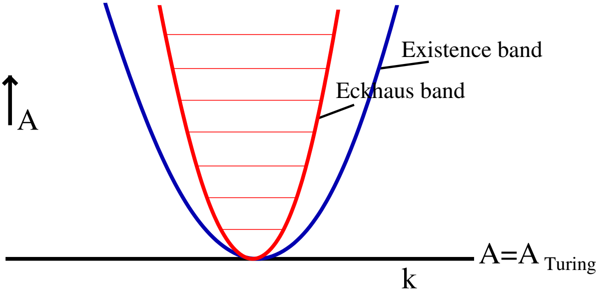


Figure: The Eckhaus band.

The Eckhaus band: nonsymmetric case ($C > 0$)

Let A be the amplitude of the emerging pattern. The evolution of A is now described by the Complex GLE :

$$A_\tau = (\alpha_1 + i\alpha_2)A_{\xi\xi} + (\beta_1 + i\beta_2)A + (\gamma_1 + i\gamma_2)|A|^2 A.$$

Spatially periodic patterns appear at a Turing-Hopf bifurcation and are now travelling:

$$A(\xi, \tau) = Re^{i(k_c x + \omega_c t)}$$

The Turing-Hopf bifurcation is subcritical if the so-called Landau coefficient satisfies

$$1 + \frac{\alpha_2\gamma_2}{\alpha_1\gamma_2} > 0.$$

Super- vs. subcritical bifurcation

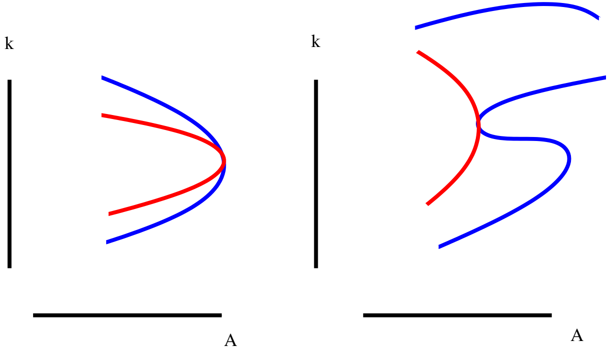


Figure: Schematic picture of supercritical bifurcation (left) and subcritical bifurcation (right).

Results: calculation of Landau-coefficient

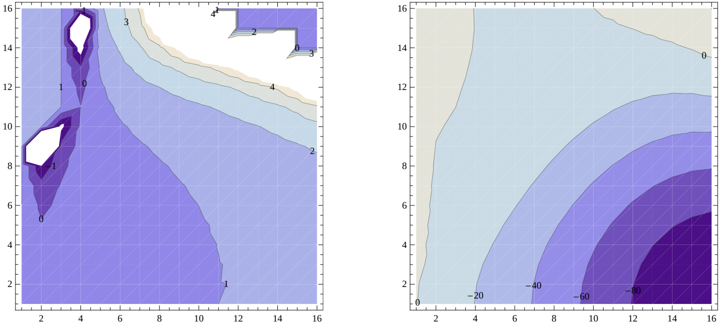


Figure: plots of b against C, for $\gamma = 1$ (supercritical) and $\gamma = 2$ (subcritical for some values of b if $C > 0$)

Busse balloon

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- No mathematical analysis possible! $\longrightarrow$ but numerics! Continuation and bifurcation software (AUTO)

Example: Busse balloon for Gray-Scott system

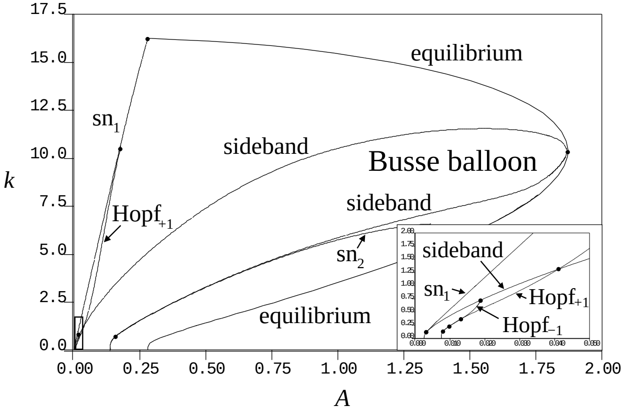


Figure: A Busse balloon for the Gray-Scott model with $b = \ldots$ and $\gamma = 1$.

The ‘Hopf dance’

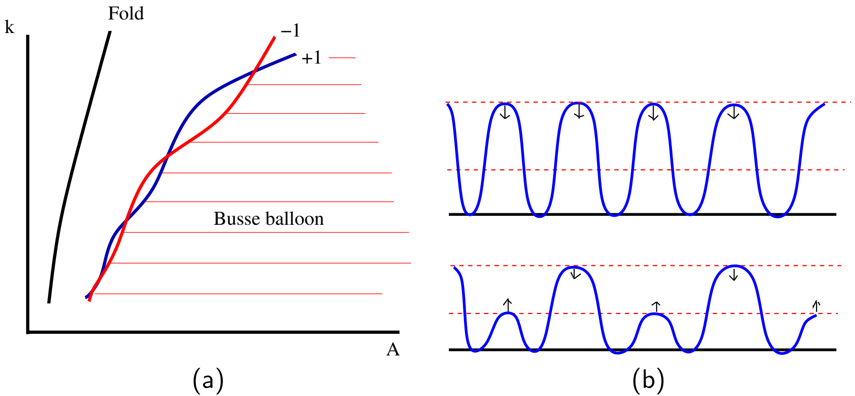


Figure: (a) The Hopf-dance enlarged, schematically. (b) Oscillations in or out of phase (i.e. for $\gamma = 0$ or $\gamma = 1$).

Spectrum of solution at Hopf instability

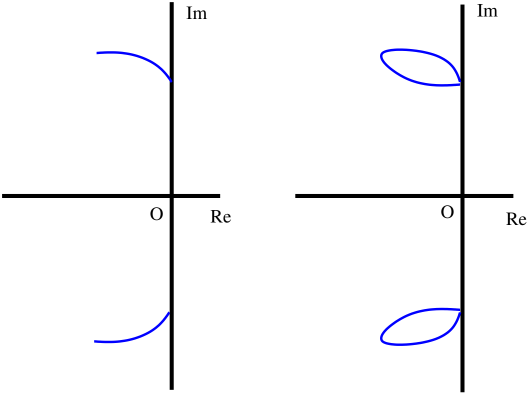
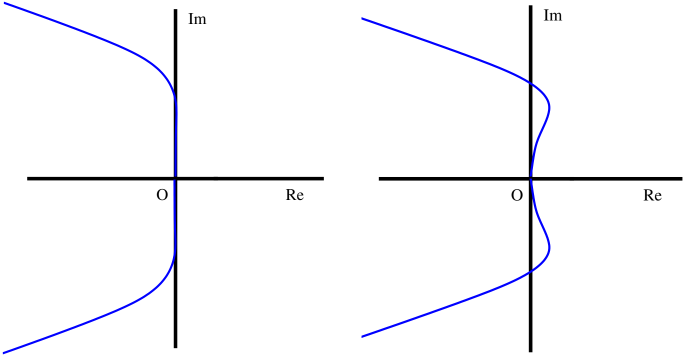


Figure: Spectrum for a solution that undergoes a Hopf instability. Left the case for reversible systems (like Gray-Scott), right the case for nonreversible systems.

Spectrum of solution at sideband instability



(a) (b)

Figure: Spectrum for a solution that undergoes a sideband instability.

Busse balloon for $C > 0$

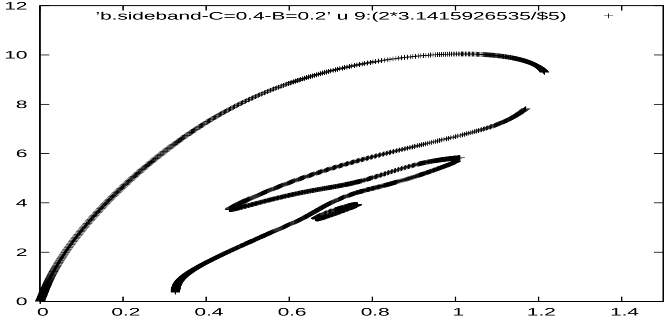


Figure: (Nonedited version of the) Busse balloon for $C = 0.4$, $B = 0.2$, $\gamma = 1$.

Disappearing Hopf dance with increasing C

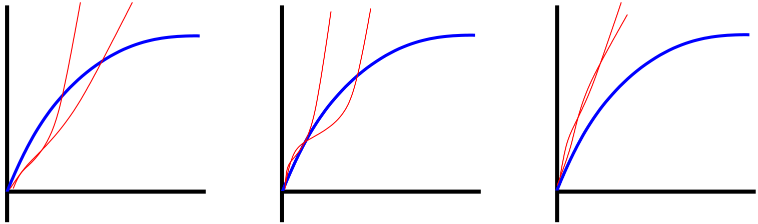


Figure: Schematic picture showing the Hopf instabilities with γ-eigenvalues for $\gamma = 0$ and $\gamma = \pi$ and the sideband instability in the neighbourhood of the origin, for $C = 0.0$ (left), $C = 0.2$ (middle) and $C = 0.4$ (right).

Conclusions

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- for $\gamma = 1$ and $C = 0$ (Gray-Scott) the boundary of the full Busse balloon consists of a complex interplay of different instabilities (Hopf, sideband, fold...)

Conclusions

- for $\gamma = 2$ and $C > 0$ there are choices for B such that the Turing-Hopf bifurcation becomes subcritical (stable patterns and stable background state)
- for $\gamma = 1$ and $C = 0$ (Gray-Scott) the boundary of the full Busse balloon consists of a complex interplay of different instabilities (Hopf, sideband, fold...)
- for $C > 0$ (and $\gamma = 1$), the Hopf dance moves out of the stable Busse region

Ongoing research

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