

An Organ Transplantation Model
and Double Matching Queues: A New Look
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Dutch-Israeli workshop

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Motivation

- A personal experience
- ABO issue
- FCFS issuing policy

The Model

Poisson stream of organs - rate λ

Poisson stream of patients - rate μ

Potential shelf life = 1

Random Patience - distribution H

$$\gamma = \int_0^{\infty} [1 - H(x)] dx$$

Functionals

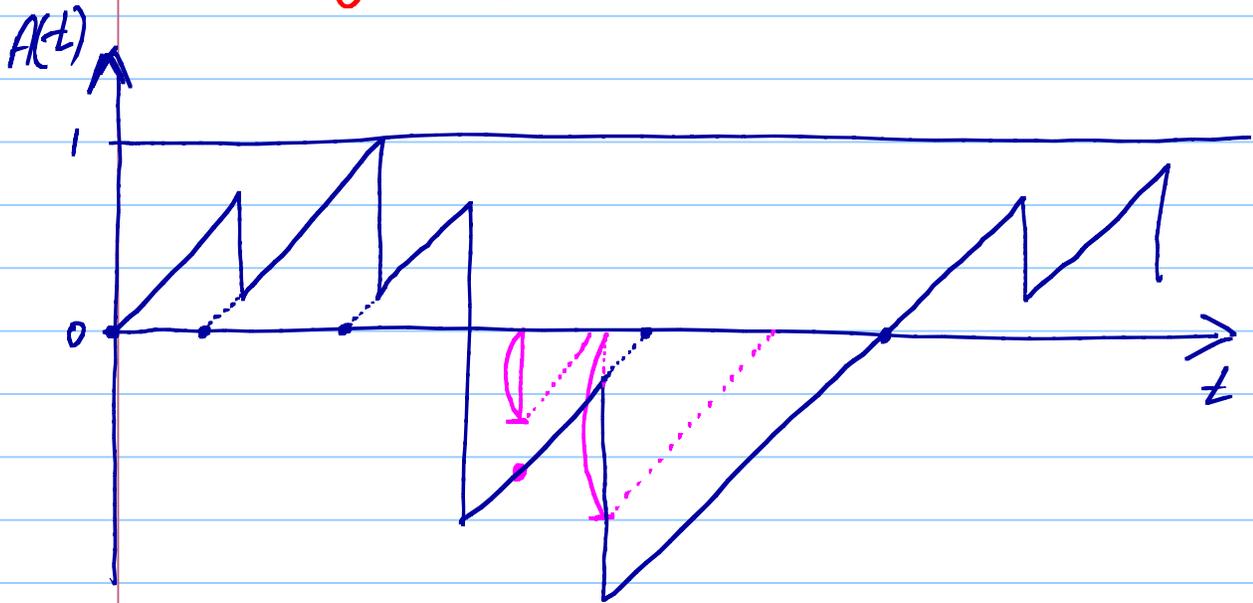
$\lambda^*(\lambda, \mu) \equiv \lambda^*$ - outdating rate

$\mu^*(\lambda, \mu) \equiv \mu^*$ - unsatisfied demand rate

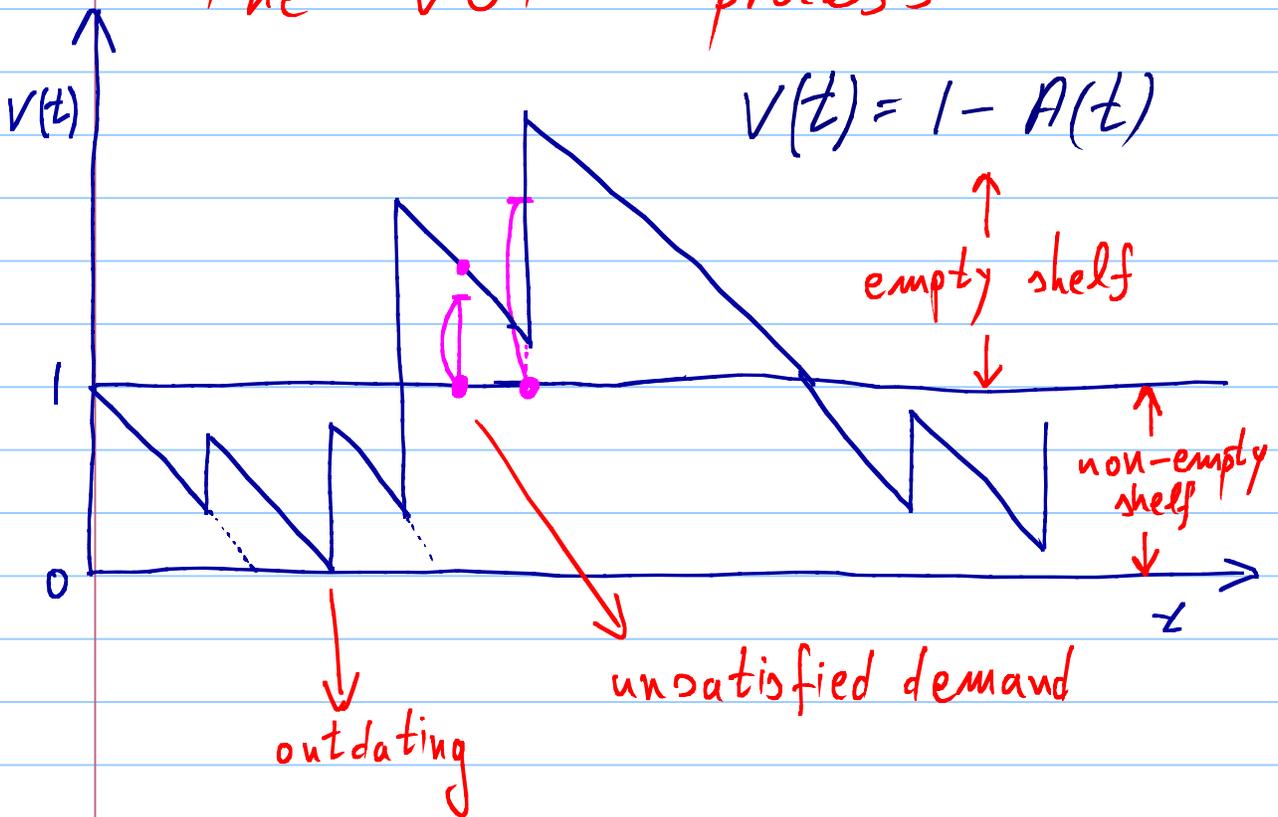
Conservation law

$$\mu - \mu^* = \lambda - \lambda^*$$

The age process

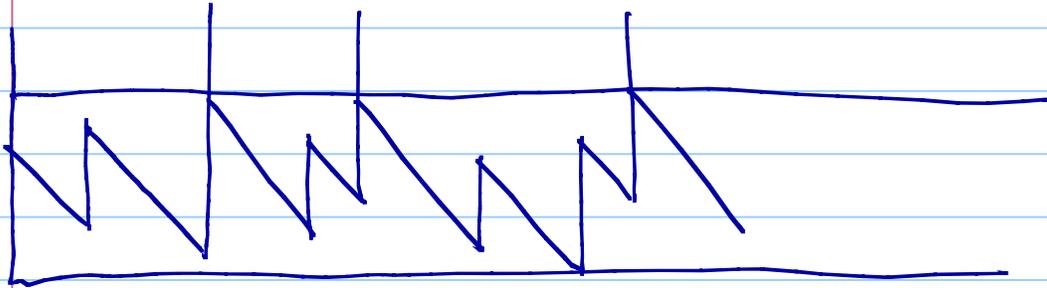


The VOP process

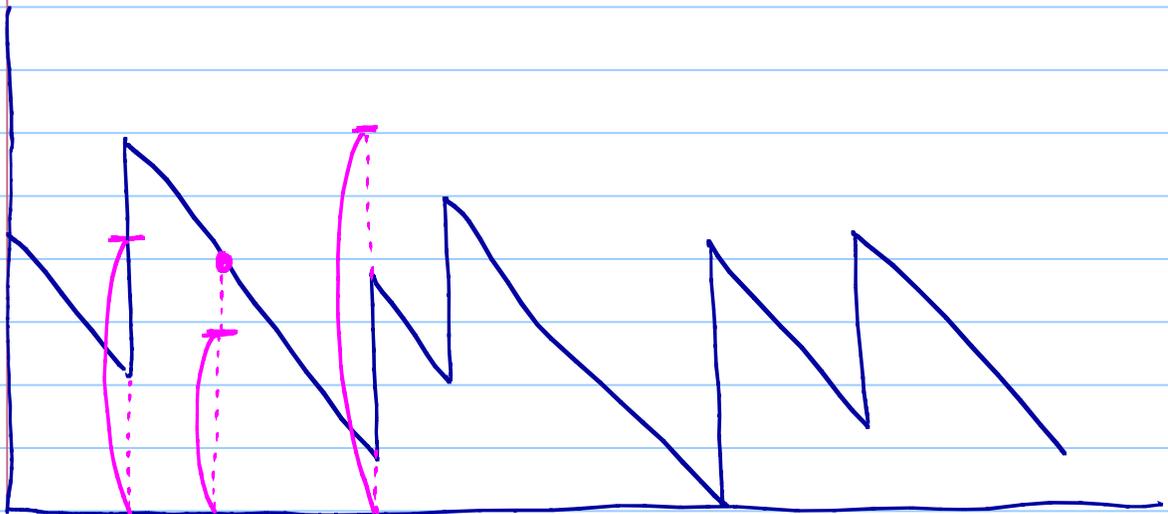


Separation

Below level 1 - Finite dam



Above level 1 - $M/G/1+G$



steady state analysis for VOP

For $x < 1$

$$f(x) = \mu \int_0^x [1 - G(x-w)] f(w) dw + f(0) [1 - G(x)]$$

For $x > 1$

$$f(x) = \mu \int_0^1 [1 - G(x-w)] f(w) dw + f(0) [1 - G(x)] \\ + \mu \int_1^x [1 - G(x-w)] [1 - H(w-1)] f(w) dw$$

Interpretations

1. The VOP is a regenerative process,
2. $F(1) = \int_0^1 f(x) dx$ — Probability that the shelf is not empty

3. $1 - F(1) = \int_1^{\infty} f(x) dx$ - Probability that the shelf is empty, but the waiting list of patients is not necessarily empty

4. $f(0)$ - rate of outdatings
 $f(0) = \lambda^*$

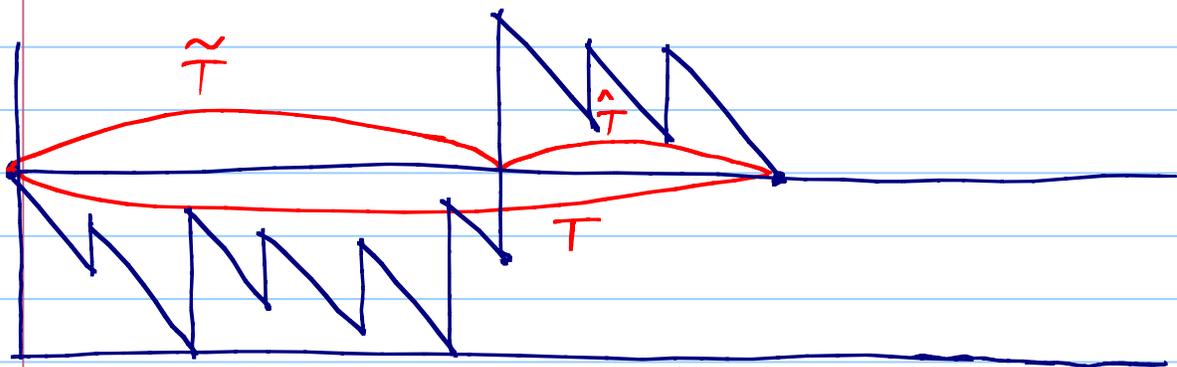
5. $f(1)$ rate of arrivals into an empty shelf

6. Let \hat{f} be the steady state density of the M/G/1+G system and \tilde{f} be the steady state density of the finite dam

$$\hat{f}(x) = \mu \int_0^x [1 - G(x-w)] [1 - H(w)] f(w) dw + f(0) [1 - G(x)] \quad x \geq 0$$

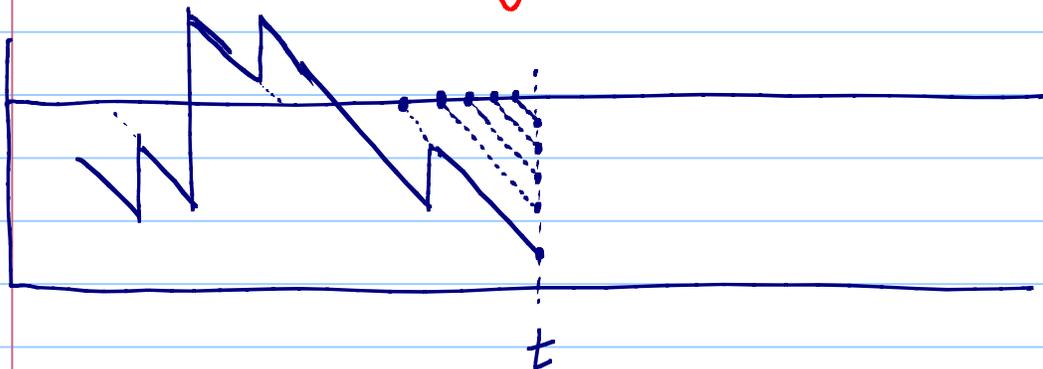
$$\tilde{f}(x) = \mu \int_0^x [1 - G(x-w)] \tilde{f}(w) dw + \tilde{f}(0) [1 - G(x)] \quad 0 \leq x \leq 1$$

Then ,



$$P(\text{empty shelf}) = \frac{E\tilde{T}}{ET} = \frac{1/\tilde{f}(1)}{1/f(1)} = \frac{f(1)}{\hat{f}(1)}$$

Number of organs on the shelf



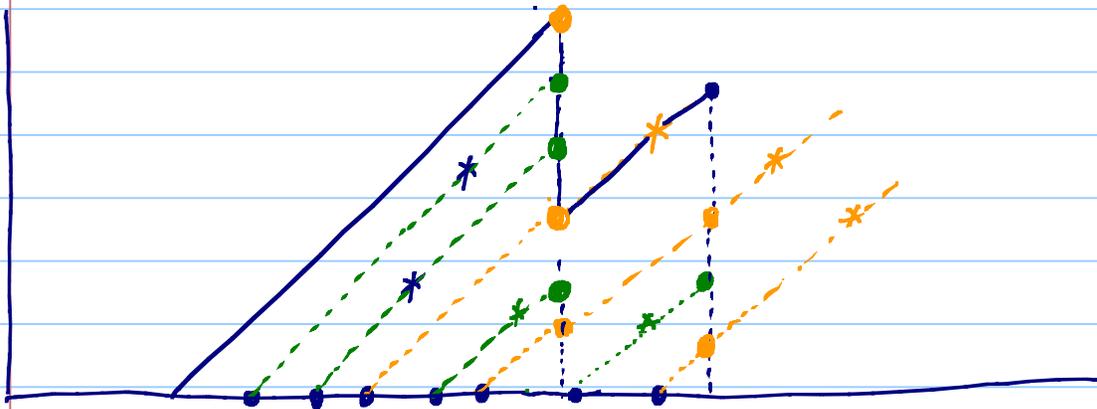
Let N_0 be the number of organs on the shelf

$$\{N_0 = n\} = \begin{cases} \{\text{empty shelf}\} & n=0 \\ \left\{ \begin{array}{l} (n-1) \text{ arrivals during} \\ \text{the age of the} \\ \text{oldest item} \end{array} \right\} & n \geq 1 \end{cases}$$

$$P(N_0 = n) = \begin{cases} 1 - F(1) & n=0 \\ \int_0^1 \frac{e^{-\lambda(1-x)} [\lambda(1-x)]^{n-1}}{(n-1)!} f(x) dx & n > 0 \end{cases}$$

$$Ez^{N_0} = 1 - F(1) + z \int_0^1 e^{-\lambda(1-z)(1-x)} f(x) dx$$

Number of waiting patients



The modified system.

In the modified system patients don't leave due to impatience. If the patience is less than the waiting time the original service requirement is changed to 0.

Result. The number of customers left behind at a moment of departure in the original system is equal to the

number of green customers left behind at time of departure in the modified system.

$$Q_n = P \left[n \text{ patients left behind after departure} \right]$$

$$= \int_0^{\infty} \sum_{k=n}^{\infty} \binom{k}{n} p_t^n q_t^{k-n} \cdot \frac{e^{-\lambda t} (\lambda t)^k}{k!} dF_{D_{mod}}(t)$$

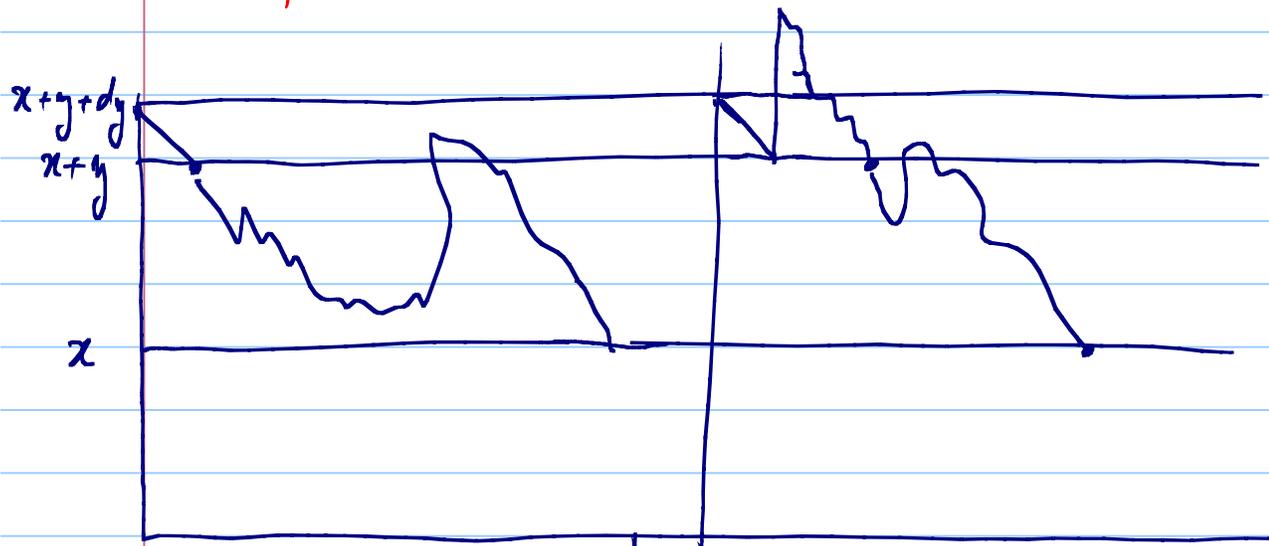
$$= \int_0^{\infty} \frac{e^{-\lambda p_t t} (\lambda p_t t)^n}{n!} dF_{D_{mod}}(t)$$

where

$$p_t = \frac{1}{t} \int_0^t [1 - H(t-s)] ds$$

$$= \frac{1}{t} \int_0^t [1 - H(s)] ds$$

Emptiness Period



$$\Gamma(\alpha; x+y+dy, x)$$

$$= [1 - \lambda(1-H(x+y))dy][1 - \alpha dy] \Gamma(\alpha; x+y, x)$$

$$+ \lambda dy(1-H(x+y)) \psi(x+y) \Gamma(\alpha; x+y, x)$$

where

$$\psi(\alpha, x) = \int_0^{\infty} \Gamma(\alpha; x+y, x) dG(y)$$

Then

$$\varphi(\alpha, x) = \int_0^{\infty} e^{-\lambda y + \lambda \int_0^y (1-H(x+u)) \{1-\varphi(\alpha, x+u)\} du} dG(y).$$