Universal Generation
of
Fractal Statistics

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Introduction

Introduction

Internet-age data traffic features “fractal statistics”:

- heavy tails – “Noah effect”
- long-ranged correlations – “Joseph effect”
- self-similarity

(Leland et. al. 1994 , Paxon & Floyd 1994 ,
Crovella & Bestavros 1996 , Willinger et. al. 1997 , ...)

Introduction

Noah effect: *"...were all the fountains of the great deep broken up, and the windows of heaven were opened. And the rain was upon the earth forty days and forty nights."* Genesis, 7: 11-12

Joseph effect: *"...there came seven years of great plenty throughout the land of Egypt. And there shall arise after them seven years of famine..."* Genesis, 41: 29-30

(Mandelbrot and Wallis 1968)

Introduction

Explaining the Joseph-effect and self-similarity: signal-superposition models yielding ***FBM*** output

Superpositions of:

- renewal processes (Mandelbrot 1969)
- on-off processes (Taqqu et. al. 1997)
- persistent random walks (Enriquez 2004)
- OU processes (Leonenko & Taufer 2005)
- . . .

Introduction

The aforementioned superposition models have:

- specific building-block processes
- specific pre-set goal – ***FBM*** output

Our goal: construct a superposition model which

- accommodates arbitrary building blocks
- generates "fractal statistics" universally, and not as a pre-set goal

The Superposition Model

$$Y(t) = \sum a_k X_k(\omega_k(t-\tau_k))$$

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- The processes $\{X_k\}_k$ are i.i.d. copies of a generic – yet arbitrary – input pattern process X
- Each process has its own random transmission parameters: amplitude a_k ; frequency ω_k ; initiation epoch τ_k
- The set of transmission parameters $\{(a_k,\omega_k,\tau_k)\}_k$ is a Poisson process with intensity $\lambda(a,\omega,\tau)$

Superposition-model Examples

$$Y(t) = \sum a_k X_k(\omega_k(t-\tau_k))$$

Conceptual Examples

Communication: Signals superposition in a transmission channel

Physics and EE: Superposition of noise sources building up a background noise

Hydrology: Superposition of flows building up the Nile flow (Hurst)

Economics: Superposition of tax revenues building up a State tax income

Conceptual Examples

Probe dynamics

Probe in a “stochastic bath”

Probe “hit” by random “gusts”

X = impact pattern of a generic gust on probe

Y = probe trajectory

Conceptual Examples

Internet Provider

{k} fixed and static set of customers

X = customer load pattern (stationary)

$\tau_k = 0$

Internet Video Server

{k} random and dynamic set of demands

X = video download pattern (dissipative)

Random Walks

Random walk (compound Poisson process)

$X(t) = 1$

$\{a_k\}_k$ i.i.d. random variables

$\{\omega_k\}_k$ constant (all = 1)

$\{\tau_k\}_k$ standard Poisson process

M/G/∞ Processes

M/G/∞ queue process

$X(t) = \mathbf{I}(0<t<1)$

$\{a_k\}_k$ constant (all = 1)

$\{\omega_k\}_k$ i.i.d. random variables

$\{\tau_k\}_k$ standard Poisson process

(Takacs 1962 , Cox & Miller 1965,

Cox & Isham 1980, …)

Shot Noise Processes

Classic Shot Noise

$X(t) = \exp(-t)$

$\{a_k\}_k$ i.i.d. random variables

$\{\omega_k\}_k$ constant relaxation rate

$\{\tau_k\}_k$ standard Poisson process

(Rice 1944 , Rice 1945, …)

Shot Noise Processes

Fractal Shot Noise

$X(t)$ = power-law impulse-response function

$\{a_k\}_k$ i.i.d. random variables

$\{\omega_k\}_k$ constant relaxation rate

$\{\tau_k\}_k$ standard Poisson process

(Lowen & Teich 1989, Lowen & Teich 1990)

Shot Noise Processes

Shot Noise with random relaxtions

$X(t) = \exp(-t)$

$\lambda(a,\omega,\tau) = \lambda_{\mathbf{amp}}(a)\cdot\lambda_{\mathbf{fre}}(\omega)$

(Eliazar & Klafter 2007)

Universal Statistics

Universal Statistics

- Let $S(Y)$ be a statistic of interest of the output process Y
- We seek Poissonian intensities $\lambda(a,\omega,\tau)$ which render the statistic $S(Y)$ invariant, up to a scale factor, with respect to the input pattern X
- The obtained statistics $S(Y)$ are termed "***universal statistics***"

Stationary Processes

Stationary Processes

- Statistic of interest:

  the process's stationary distribution

- <u>Theorem</u>: universal statistics are ***Levy laws***

$$\mathbf{E}\left[\exp(i\theta Y(t))\right] = \exp(-c_X \cdot |\theta|^{\alpha}) \qquad (0<\alpha<2)$$

- <u>Conclusion</u>: universality of the ***Noah effect***

Stationary Processes

- Statistic of interest:

  the process's power spectrum

- <u>Theorem</u>: universal statistics are ***1/f noises***

$$\mathbf{Spec}_Y(f) = c_X / |f|^{\alpha} \quad (0<\alpha<1)$$

- <u>Conclusion</u>: universality of the ***Joseph effect***

Non-stationary Processes

Non-stationary Processes

- Statistic of interest:

  the process's mean-square-displacement

- <u>Theorem</u>: universal statistics are ***power-laws***

$$\mathbf{Var}\left[Y(t)\right] = c_X \cdot t^{\alpha} \qquad (\alpha>0)$$

- <u>Conclusion</u>: universality of ***anomalous diffusion***

Non-stationary Processes

- Statistic of interest:

  the process's power spectrum

- <u>Theorem</u>: universal statistics are ***1/f noises***

$$\mathbf{Spec}_Y(f) = c_X / |f|^{\alpha} \quad (0<\alpha<2)$$

- <u>Conclusion</u>: universality of ***pink noise***

Ultra Diffusion

Ultra Diffusion

- Diffusion model: Brownian motion
- Variance characterization of Brownian motion

$$\mathbf{Var}\left[X(t)\right] = D \cdot t$$

- Fourier characterization of Brownian motion

$$\mathbf{E}\left[\exp\left(i\theta X(t)\right)\right] = \exp\left(-D \cdot t \cdot \theta^2/2\right)$$

Ultra Diffusion

- Fourier characterization does not require a finite-variance condition to hold
- Definition of ultra diffusion

$$\mathbf{E}\left[\exp\left(i\theta X(t)\right)\right] = \exp\left(-c\cdot\psi(t)\cdot\phi(\theta)\right)$$

- In case of finite variance

$$\mathbf{Var}\left[X(t)\right] = \left(c\phi''(0)\right)\cdot\psi(t)$$

Ultra Diffusion

Examples of ultra diffusion processes:

- Poisson and compound Poisson processes
- Brownian motion and FBM
- Levy processes and FSLM
- OU processes driven by stable Levy noises
- M/G/∞ processes

Ultra Diffusion: Universal Statistics

- Statistics of interest:

Temporal function $\psi(t)$ and Fourier function $\phi(\theta)$

- <u>Theorem</u>: universal statistics are ***power-laws***

$$\psi(t) = t^{\alpha} \quad , \quad \phi(\theta) = |\theta|^{\beta} \qquad (\alpha>0 \, , \, 0<\beta<2)$$

- <u>Conclusion</u>: joint universality of ***anomalous diffusion*** and of ***Levy laws*** – the ***Noah effect***

Self-Similarity

Self-Similarity

Definition of self-similarity

A stochastic process $Y(t)$ $(t \geq 0)$ is self-similar with Hurst exponent H if the scaled processes

$$Y(ct) \text{ and } c^H Y(t) \qquad (t \geq 0)$$

are equal in law (for any positive scale c)

Self-Similarity

Universality theorem for self-similarity

The output process Y is self-similar, with Hurst exponent *H*, for all inputs X, if and only if the Poissonian intensity satisfies the scaling relation:

$$\lambda(c^{H}a,\omega/c,c\tau) = c^{-H}\cdot\lambda(a,\omega,\tau)$$

(the scaling relation holding for all c>0)

Self-Similarity

Self-similarity corollaries

If the output process Y is of finite-variance then

- ***Anomalous diffusion*** statistics are displayed:

$$\mathbf{Var}\left[Y(t)\right] = c_X \cdot t^{2H}$$

- ***1/f noise*** statistics are displayed:

$$\mathbf{Spec}_Y(f) = c_X / |f|^{1+2H}$$

Self-Similarity of Propagating Populations

$$Y_k(t) = a_k X_k(\omega_k(t-\tau_k))$$

- Propagating particles launched from the origin of the *d*-dimensional Euclidean space
- The trajectory of particle k is the stochastic process Y_k
- The random processes $\{X_k\}_k$ and the random parameters $\{(a_k,\omega_k,\tau_k)\}_k$ are as before

The Displacement Ensemble

- Displacement of particle k at time t:

$$D_k(t) = |Y_k(t)|$$

- Displacement ensemble at time t:

$$\boldsymbol{D}(t) = \{ D_k(t) \; ; \; \tau_k \leq t \}$$

- Displacement ensemble is a Poisson process with intensity

$$\lambda_{\boldsymbol{D}(t)}(x) \quad (x>0)$$

The FPT Ensemble

- First Passage Time (FPT) of particle k:

$$T_k(r) = \inf\{ t \geq 0 : |Y_k(t)| \geq r \}$$

- FPT ensemble of radius r:

$$\boldsymbol{T}(r) = \{ T_k(r) \}_k$$

- FPT ensemble is a Poisson process with intensity

$$\lambda_{\boldsymbol{T}(r)}(x) \quad (x>0)$$

Self-Similarity

- Statistical similarity of Poisson processes: intensities equal up to a multiplicative factor
- Self-similarity of the displacement ensembles:

$$\lambda_{\boldsymbol{D}(t)}(x) = c_{Dis} \cdot M_{Dis}(t) \cdot \Lambda_{Dis}(x)$$

- Self-similarity of the FPT ensembles:

$$\lambda_{\boldsymbol{T}(r)}(x) = c_{FPT} \cdot M_{FPT}(r) \cdot \Lambda_{FPT}(x)$$

Displacements: Universal Statistics

- Statistics of interest:

  Intensity functions $M_{Dis}(t)$ and $\Lambda_{Dis}(x)$

- <u>Theorem</u>: universal statistics are ***power-laws***

$$M_{Dis}(t) = t^{\alpha} \ , \ \Lambda_{Dis}(x) = x^{-\beta-1} \qquad (\alpha,\beta>0)$$

- <u>Conclusion</u>: universal statistics of the maximal displacements are ***Frechet laws***

FPTs: Universal Statistics

- Statistics of interest:

  Intensity functions $M_{FPT}(r)$ and $\Lambda_{FPT}(x)$

- <u>Theorem</u>: universal statistics are ***power-laws***

$$M_{FPT}(r) = r^{-\beta} \ , \ \Lambda_{Dis}(x) = x^{\alpha-1} \qquad (\alpha,\beta>0)$$

- <u>Conclusion</u>: universal statistics of the minimal FPTs are ***Weibull laws***

Conclusions

Conclusions

- The ***superposition model*** is a general and robust model serving as a unified framework for the ***universal generation*** of ***fractal statistics***: Statistics possessing an intrinsic ***power-law*** structure.
- The universal power-law structures obtained are <u>not</u> pre-set goals! Rather, they ***emerge*** via the ***invariance***-requirement with respect to the arbitrary input patterns.

Conclusions

The universal fractal statistics obtained include:

- ***Lévy laws*** – the ***Noah effect***
- ***1/f noises*** – the ***Joseph effect***, ***pink noise***
- ***Anomalous*** diffusion and ultra diffusion
- Statistically ***self-similar*** processes
- ***Power-law*** Poisson processes
- Extreme value laws – ***Frechet*** and ***Weibull***

References

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The End

Thank you