A trivia question

What is in common between the following locations:

Eindhoven
Leiden
Sydney
Amsterdam
Hadera
Haifa
France
Shfayim

DROS

Discriminatory random order service:

- Each customers possesses a parameter p_i
- Upon service commencement (no preemption), customer i enter service with probability $p_i/\Sigma_j p_j$.

Haviv and van der Wal 1997: M/M/1, parameter x, costs x. What is the equilibrium purchasing strategy?

Answer: pure strategy. Pay

$$\frac{C\rho^2}{\mu(1-\rho)(2-\rho)}$$

DPS

Similar result for DPS: In DROS lotteries at service commencements, in DPS it is at service completions.

Still open: Equilibrium payment in case of M/G/1? (for both DROS and DPS)

M/G/1 with relative priority

Class i: λ_i, $\overline{x}_i$, $\overline{x^2}_i$, p_i.

Mean value analysis: Haviv and Van der Wal (2008).

The same if HOL is assumed among classes.

Higher moments: A paper by .... (under review).

Higher moments in case of HOL?

When to arrive at a queue with tardiness costs?

Moshe Haviv

The Hebrew University of Jerusalem

Eindhoven, 30.09.10

Concert hall with early birds

- gate opens at zero and closes at T
- FCFS, inclusive of early birds
- all arrivals prior to T are served
- single server
- service exp(μ)
- N= no. of arrivals Poisson(λ)
- α= cost per unit of queueing
- $\beta > 0$= cost per unit of tardiness (from zero)

Equilibrium

Symmetric (Nash) equilibrium: an arrival strategy (mixing is possible), if used by all, nobody has an incentive to do otherwise

Hassin and Glazer (1983): $\beta = 0$ and T finite

Jain, Juneja and Shimkin, (2010): Fluid approximation, $T = \infty$

Juneja and Shimkin (2010+): $\beta > 0$, $T = \infty$, any distribution for N

Equilibrium, $\beta > 0$, $T = \infty$

- not a pure strategy
- mixed strategy but without atoms
- mixed strategy with a positive density along an interval
- the arrival interval $[-w, T_e]$
- uniform density along $[-w, 0)$
- continuous density but not at zero (downwards)

Assume $\beta > 0$: Otherwise, $T_e = \infty$ and zero waiting costs

Equilibrium conditions, $T = \infty$

$f(t)$: density of the arrival strategy

$$\int_{-w}^{T_e} f(t)\,dt = 1$$

$f(t)$ determines the queueing process

$w(t) =$mean queueing time if arrive at t

Equilibrium conditions:

$$(\alpha + \beta)w(t) + \beta t = \text{Constant}, \quad -w \leq t \leq T_e$$

$$(\alpha + \beta)w(t) + \beta t \geq \text{Constant}, \quad t < w, t > T_e$$

Reverse engineering: Find w, T_e and $f(t)$ such that the equilibrium conditions hold

Equilibrium

Equilibrium:

$$f(t) = \frac{\mu}{\lambda}\frac{\alpha}{\alpha+\beta}, \quad -w \le t < 0$$

$f(t)$ is discontinuous at $t = 0-$

$$\int_0^{T_e} f(t)\,dt = 1 - w\frac{\mu}{\lambda}\frac{\alpha}{\alpha+\beta}$$

Initial conditions:

$$P_k(0) = e^{-w\mu\frac{\alpha}{\alpha+\beta}} \frac{(w\mu\frac{\alpha}{\alpha+\beta})^k}{k!}, \quad k \geq 0$$

Equilibrium:

$$f(t) = \frac{(1-P_0(t))\mu}{\lambda} - \frac{\beta\mu}{(\alpha+\beta)\lambda}, \quad 0 \leq t \leq T_e$$

Dynamics:

$$P_0'(t) = P_1(t)\mu - P_0(t)\lambda f(t), \quad 0 < t < T_e$$

$$P_k'(t) = P_{k-1}(t)\lambda f(t) + P_{k+1}(t)\mu - P_k(t)(\lambda f(t) + \mu), \quad 0 < t < T_e, \ k \geq 1$$

Equilibrium:

$$\alpha(1 - P_0(T_e)) = \beta P_0(T_e) \quad (\text{or} \quad f(T_e) = 0)$$

Equilibrium, $T < \infty$

- If $T > T_e$, as $T = \infty$
- If $T < T_e$, replace T_e with T and ignore the last condition

$$\alpha(1 - P_0(T_e)) = \beta P_0(T_e)$$

In fact,

$$\alpha(1 - P_0(T)) > \beta P_0(T)$$

Social cost: $\lambda \alpha w$

Concert hall w/o early birds

- gate opens at zero and closes at T
- FCFS, exclusive of early birds
- early birds enter at random
- all arrivals prior to T are served
- single server
- service exp(μ)
- N= no. of arrivals Poisson(λ)
- α= cost per unit of queueing
- $\beta > 0$= cost per unit of tardiness (from zero)

Hassin and Kleiner (2010): $\beta = 0$, T finite

Equilibrium

1. if $T \leq T_1$, pure: arrive at zero
   $T_1 = \infty$ is possible
2. if $T_1 < T \leq T_e$,
   - atom at zero
   - positive density along $[t', T]$
3. if $T > T_e$
   - atom at zero
   - positive density along $[t', T_e]$

Equilibrium

N_p Poisson(λp)
X_i, iid, exp(μ)

$$g(t) = (\alpha + \beta)\mathsf{E}(\sum_{i=0}^{N_1} X_i - t)^+ + \beta t, \quad t \geq 0$$

$$t^* = \arg\min_{t \geq 0} g(t)$$

If $g(t^*) \geq \lambda(\alpha + \beta)/2\mu$

$$\Downarrow$$

Pure equilibrium: arrive at $t = 0$ (for any T)

Equilibrium, $T < \infty$

Assume $g(t^*) \leq \lambda(\alpha+\beta)/2\mu$

T_1= the smallest (among two) t such that

$$g(t) = (\alpha+\beta)\frac{\lambda}{2\mu}$$

T_e= the latest time to arrive in equilibrium when $T = \infty$ (needs to be determined).

The shape of the equilibrium depends if

- $T \leq T_1$ (pure), or
- $T_1 \leq T \leq T_e$ (mixed), or
- $T \geq T_e$ as for T_e

Equilibrium

If $T \leq T_1 \Rightarrow$ pure strategy: arrive at $t = 0$

If $T_1 \leq T \leq T_e$,

- atom of size p_0 at zero
- zero density along $(0, t')$
- positive density along $[t', T]$

$$(\alpha + \beta)\frac{\lambda p_0}{2\mu} = (\alpha + \beta)\mathsf{E}(\sum_{i=0}^{N_{p_0}} X_i - t')^+ + \beta t',$$

One dimensional search for p_0 based on

$$\int_{t'}^{T} f(t)\,dt = 1 - p_0$$

Equilibrium

If $T \geq T_e$ as in T_e.

Finding T_2:

- For any $T \in (T_1, T_e)$, $\alpha(1 - P_0(T) > \beta P_0(T)$: A bit after T is a better response (yet, not feasible)
- T_e is the smallest T with $\alpha(1 - P_0(T)) = \beta P_0(T)$

The social cost=$\lambda \alpha w$.

Fluid approximation

- A mass of water of size Λ
- rate of service μ units of water per unit of time
- each drop needs to decide when to arrive

With early birds, fluid, $T \geq \Lambda/\mu$

Jain, Juneja and Shimkin, 2010

Equilibrium:

- uniform arrival along $[-\Lambda\beta/(\mu\alpha), \Lambda/\mu]$, rate $\mu\frac{\alpha}{\alpha+\beta}$
- social cost $= \Lambda^2\beta/\mu$ (no α)

Social optimization:

- uniform arrival along $[0, \Lambda/\mu]$, rate μ
- social cost $\Lambda^2\beta/2\mu$ (no waiting)

PoA=2, constantly

With early birds, fluid, $T < \Lambda/\mu$

Equilibrium:

- Shift all to the left, make T the upper end of the arrival interval
- Social cost: $\Lambda(\Lambda(\alpha + \beta) - \alpha\mu T)/\mu$

Social optimization:

- Arrive with rate μ along $[0, T]$. The rest at T.

Without early birds, fluid

Social optimization: As with early birds

Equilibrium:

- **If** $\beta > \alpha$**,**
  - pure strategy: arrive at 0
  - social cost: $\Lambda^2(\alpha + \beta)/2\mu$ (an improvement)
  - PoA=$(\alpha + \beta)/\beta$
- **If** $\beta \leq \alpha$**,**
  - an atom of $2\beta/(\alpha + \beta)$ at zero
  - a gap along $(0, \Lambda\beta/(\alpha\mu))$ (length as early birds horizon)
  - constant rate of $\mu\alpha/(\alpha + \beta)$ along $[\Lambda\beta/(\alpha\mu), \Lambda/\mu]$
  - social cost $\Lambda^2\beta/\mu$ (as with early birds)
  - PoA$= 2$

Why Poisson?

- huge potential arrivals n
- each comes with a tiny probability p
- number of arrivals is Poisson with mean np

An external inspector believes that the number of arrivals is Poisson

Each arrival believe the same (and same parameter) regarding the number of other arrivals

Common prior

Suppose

$$p_k = \frac{(k+1)q_{k+1}}{m}, \quad k \geq 0.$$

$q_k \geq 0$, $k \geq 0$, $\Sigma_{k=0}^{\infty} q_k = 1$, $m = \Sigma_{k=0}^{\infty} k q_k$

The p_k's are the (common) posterior of the (common) prior q_k's

Common prior

Up to the choice of q_0, any nonnegative distribution is a posterior of a unique prior

Families closed under posterior operation:

- Poisson$(\lambda) \Rightarrow$ Poisson(λ). Unique!
- Binomial$(n, p) \Rightarrow$ Binomial$(n-1, p)$
- Negative binomial$(n, p) \Rightarrow$ Negative binomial$(n+1, p)$

Thank You