September 30, 2010
Second Israeli-Dutch Workshop on Queueing Theory
EURANDOM
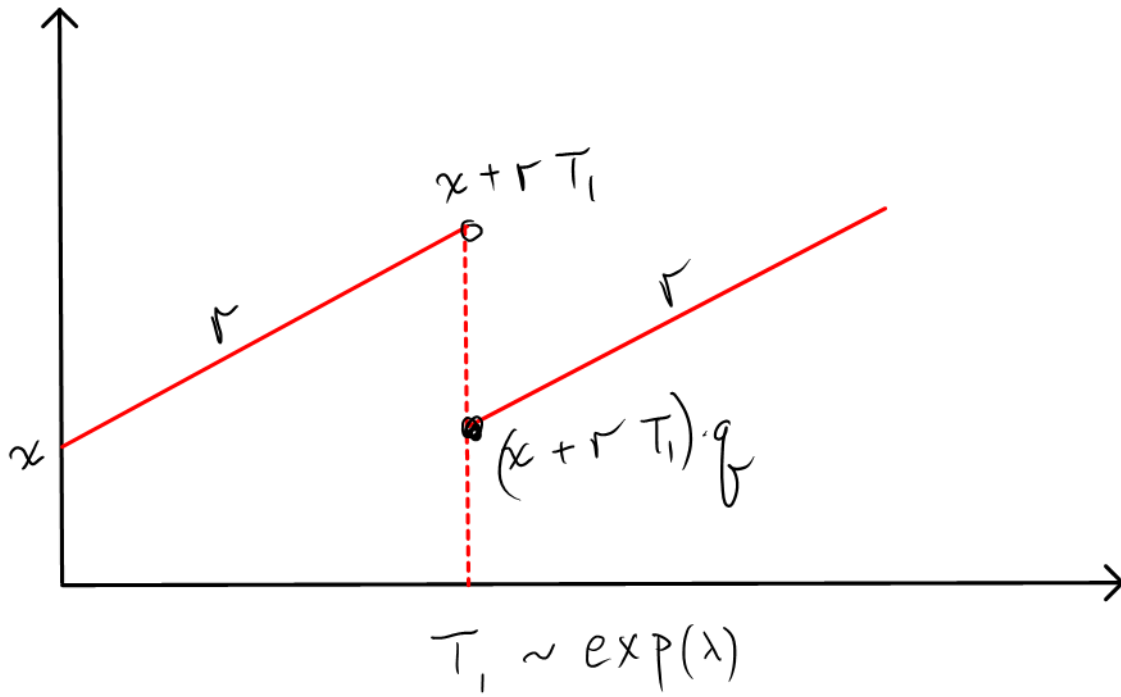
Some recent observations regarding growth-collapse processes and their generalizations
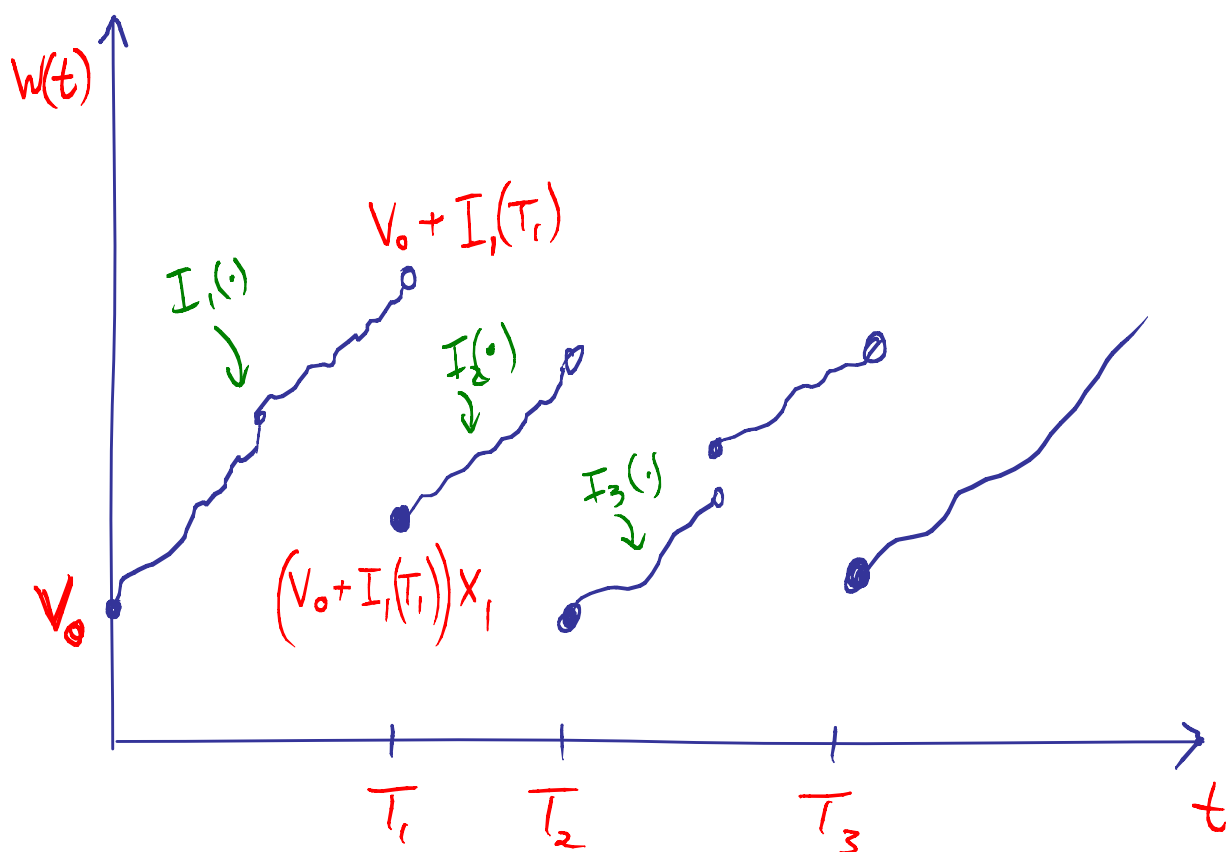
Offer Kella

The Hebrew University of Jerusalem

- **K** - 2009
  J. Appl. Probab., **46**, 363-371.

- **K** and Löpker - 2010
  Prob. Eng. Inf. Sci., **24**, 99-107.

- **K** and Yor - 2010
  Ann. Appl. Prob., **20**, 367-381.

- Boxma, **K** and Perry - 2010
  Submitted.

Simplest growth collapse process

$x + r T_1$

r

r

$(x + r T_1) \cdot q$

x

$T_1 \sim \exp(\lambda)$

Generalization

$$\bar{\tau}_n = T_n - T_{n-1}$$

$$Y_n = I_n(\bar{\tau}_n)$$

$$V_n = \left(V_{n-1} + Y_n\right)X_n$$

$$N(t) = \text{sup}\{n \mid T_n \leq t\}$$

$$W(t) = V_{N(t)} + I_{N(t)+1}\left(t - T_{N(t)}\right)$$

$$V_n = (V_{n-1} + Y_n)X_n$$

and thus

$$V_n = V_0 \prod_{j=1}^{n} X_j + \sum_{i=1}^{n} Y_i \prod_{j=i}^{n} X_j \, .$$

Level right before the nth collapse:

$$U_n = V_{n-1} + Y_n = X_{n-1}U_{n-1} + Y_n$$

Recall

$$W(t) = V_{N(t)} + I_{N(t)+1}(t - T_{N(t)})$$

Theorem 1 *Assume that $\{(X_n, Y_n) | \ n \geq 1\}$ is a stationary sequence with $EY_1 < \infty$, that*

$$\prod_{i=1}^{\infty} X_i = 0 \ , \text{a.s.} \tag{1}$$

and that its two sided extension $\{(X_n, Y_n) | \ n \in \mathbb{Z}\}$ satisfies

$$\limsup_{n \to \infty} \left(\prod_{i=-n}^{-1} X_i \right)^{1/n} \leq \rho < 1 \ , \text{a.s.} \tag{2}$$

Then $\{V_n | \ n \geq 0\}$ has a stationary version $\{V_n^* | \ n \geq 1\}$ with $P[V_n^* < \infty] = 1$ and $V_n - V_n^* \to 0$ a.s. for any initial V_0.

Theorem 2 *Assume that $\{(T_n, K_n)|\ n \geq 1\}$ is an ergodic event stationary marked point process with marks*

$$K_n = (X_n, \{I_n(t)|\ t \geq 0\})$$

as well as $E\tau_1 < \infty$, $EX_1 < 1$ and $EY_1 = EI_1(\tau_1) < \infty$. Then $\{W(t)|\ t \geq 0\}$ has a stationary version and for every function f which is bounded and Lipschitz continuous on $[0, \infty)$, we have that

$$\frac{1}{t} \int_0^t f(W(s))ds \rightarrow \frac{1}{E\tau_1} E \int_0^{\tau_1} f(V_0^* + I_1(s))ds \tag{3}$$

a.s. as $t \rightarrow \infty$. ~~Consequently,~~ $W(t)$ converges in distribution to the stationary marginal for any initial $W(0)$.

$$\frac{1}{t}\int_0^t e^{-\alpha W(s)}ds \;\rightarrow\; Ee^{-\alpha W^*(0)}$$

$$= \; E\left(e^{-\alpha V_0^*}\frac{1}{E\tau_1}\int_0^{\tau_1}e^{-\alpha I_1(s)}ds\right)$$

Corollary 1 *If in addition to the conditions of Theorem 2, $\{(\tau_n, k_n)\mid n \geq 1\}$ is i.i.d., then*

$$W \sim V + I_e \tag{4}$$

where V and I_e are ind., $W \sim W^*(0)$, $V \sim V_0^*$ and I_e is a r.v. having the dist.

$$P[I_e \in A] = \frac{1}{E\tau_1}E\int_0^{\tau_1}1_A(I_1(s))ds. \tag{5}$$

Moreover, if τ_1 and $\{I_1(t)\mid t \geq 0\}$ are ind., then $I_e \sim I_1(\tau_e)$, where

$$\tau_e \sim F_e(\square) = \int_0^{\square}\frac{P[\tau_1 > s]}{E\tau_1}ds.$$

clearing process

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Corollary 2 *If in addition to Corollary 1 we assume that X_1, $\{I_1(t)|\ t \geq 0\}$ and τ_1 are independent, then denoting $\tilde{Z}(\alpha) = Ee^{-\alpha Z}$ and $F_Z(x) = P[Z \leq x]$ for some nonnegative random variable Z we have that*

$$\tilde{V}(\alpha) = \int_{[0,1]} \tilde{V}(\alpha x)\tilde{Y}(\alpha x)F_X(dx)\,. \quad (6)$$

In particular, if $P[X = q] = 1$ for some $0 < q < 1$ then

$$\tilde{V}(\alpha) = \tilde{V}(q\alpha)\tilde{Y}(q\alpha) = \prod_{i=1}^{\infty} \tilde{Y}(q^i\alpha)\,. \quad (7)$$

Explicit computation of the stationary distribution of V:

Vervaat - 1979 (AAP **11**, 750-783).

The following seems to be new:
$X_n \sim \mathrm{Beta}(\alpha_1, \alpha_2)$, $Y_n \sim \mathrm{Gamma}(\alpha_2, \beta)$.
If $V_{n-1} \sim \mathrm{Gamma}(\alpha_1, \beta)$ then

$$(V_{n-1} + Y_n, X_n) \sim \left(V_{n-1} + Y_n, \frac{V_{n-1}}{V_{n-1} + Y_n} \right)$$

and thus

$$V_n = (V_{n-1} + Y_n) X_n$$

$$\sim (V_{n-1} + Y_n) \frac{V_{n-1}}{V_{n-1} + Y_n} = V_{n-1}$$

so in this case $\mathrm{Gamma}(\alpha_1, \beta)$ is the unique stationary distribution.

Moments for the i.i.d. case:

Since $W \sim V + I_e$,

$$EW^n = \sum_{k=0}^{n} \binom{n}{k} EV^k EI_e^{n-k}$$

Also

$$EV^n = EX^n \sum_{k=0}^{n} \binom{n}{k} EV^k EY^{n-k}$$

and thus

$$EV^n = \frac{EX^n}{1 - EX^n} \sum_{k=0}^{n-1} \binom{n}{k} EV^k EY^{n-k}$$
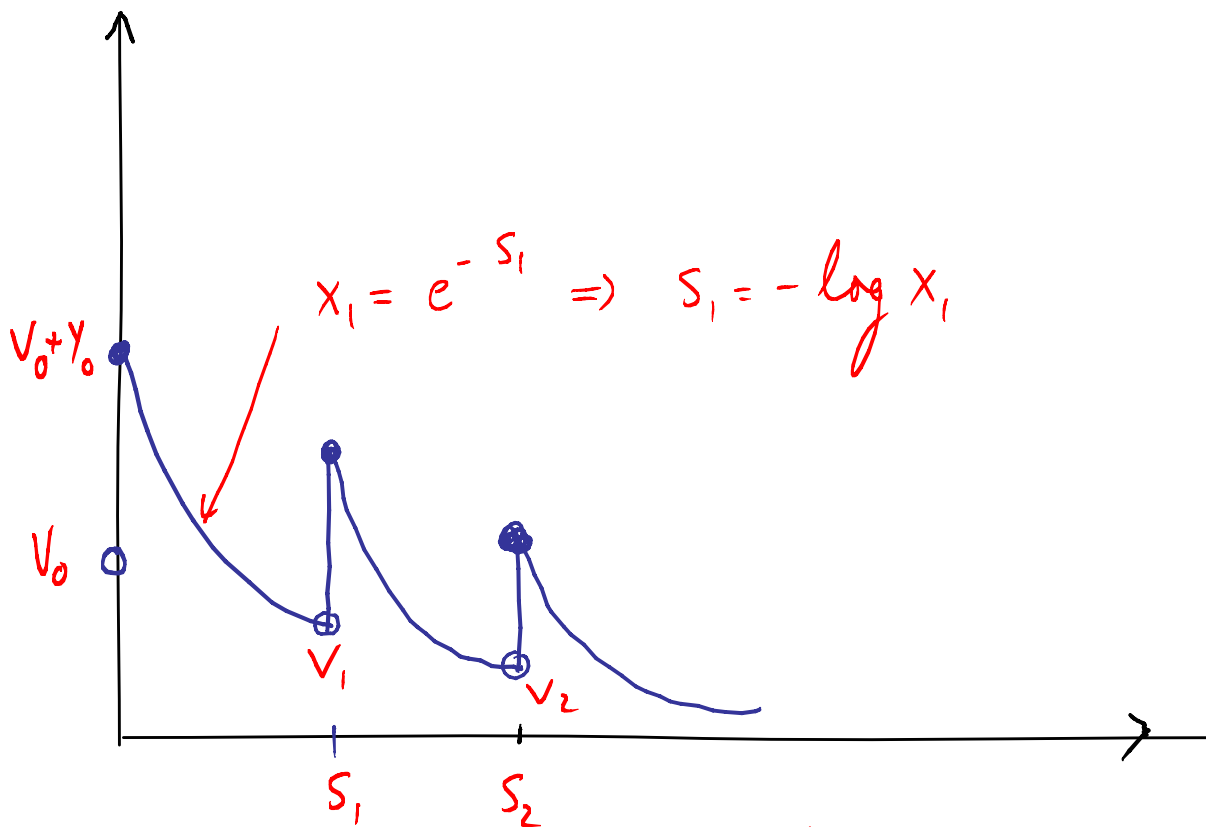
in particular

$$EV = \frac{EX}{1 - EX} EY$$

and

$$EV^2 = \frac{EX^2}{1 - EX^2} \left(EY^2 + 2\frac{EX}{1 - EX}(EY)^2 \, . \right)$$

For the case of a subordinator $EY = \eta'(0)E\tau_1$ and

$$EY^2 = -\eta''(0)E\tau_1 + (\eta'(0))^2 E\tau_1^2 \, .$$
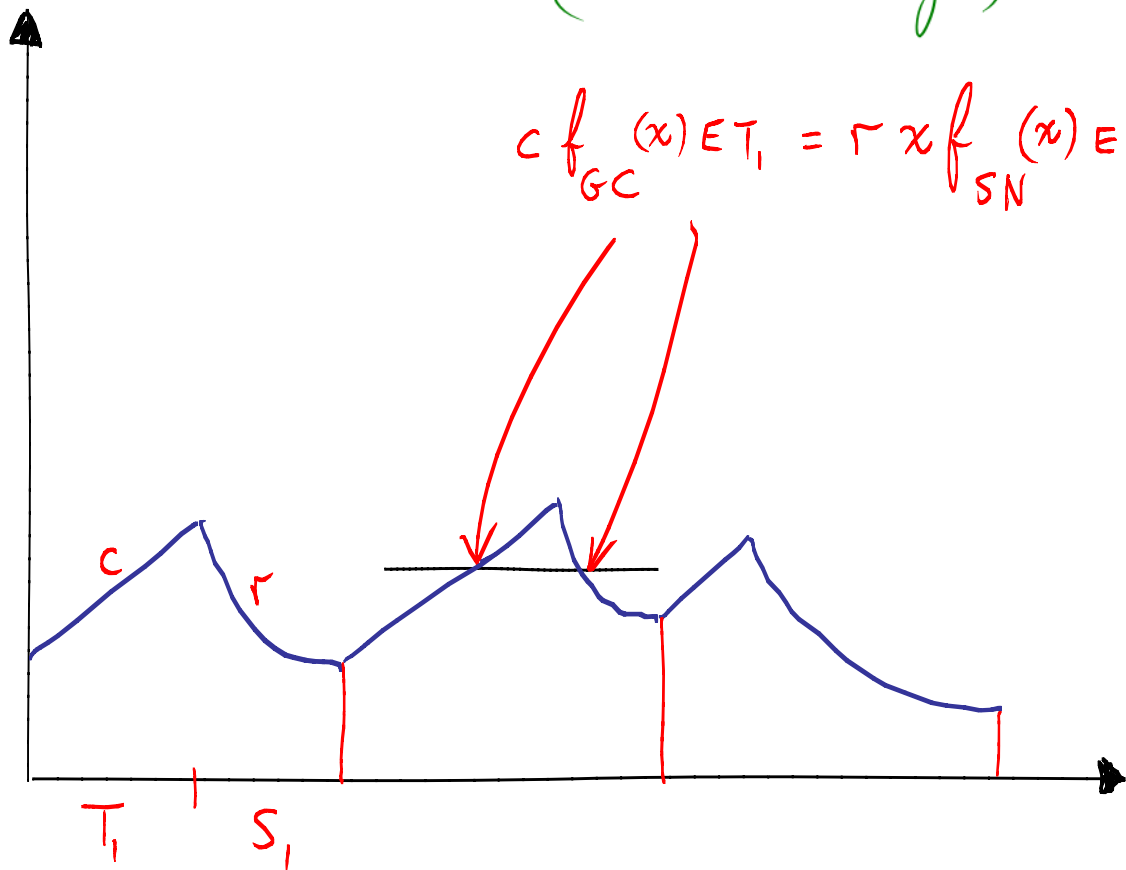
Shot-noise , linear dam

$$X_1 = e^{-S_1} \implies S_1 = -\log X_1$$

$V_0 + Y_0$

V_0

V_1

V_2

S_1

S_2

$$X_t = V_0 + \sum_{i=1}^{N(t)} Y_i - r \int_0^t X_s \, ds$$

$$N_t = \sup\{n \mid T_n \leq t\}$$

$$X_t = V_0 e^{-rt} + \sum_{i=1}^{N_t} Y_i e^{-r(t - T_i)}$$

(level crossings)

$$c f_{GC}(x) E T_1 = r x f_{SN}(x) E S_1$$

c

r

T_1 S_1

Relation with shot noise processes:

Kaspi, **K** and Perry. - 1997
(QUESTA **24**, 37-57).

$$cf_{\mathrm{gc}}(x)ET_1 = rxf_{\mathrm{sn}}(x)ES_1 \ .$$

Therefore, if for $i = \mathrm{gc}, \mathrm{sn}$ and $\alpha \geq 0$,

$$\psi_i(\alpha) = \int_0^\infty e^{-\alpha x} f_i(x)dx \ ,$$

then

$$c\psi_{\mathrm{gc}}(\alpha)ET_1 = -r\psi'_{\mathrm{sn}}(\alpha)ES_1$$

~~therefore,~~ and

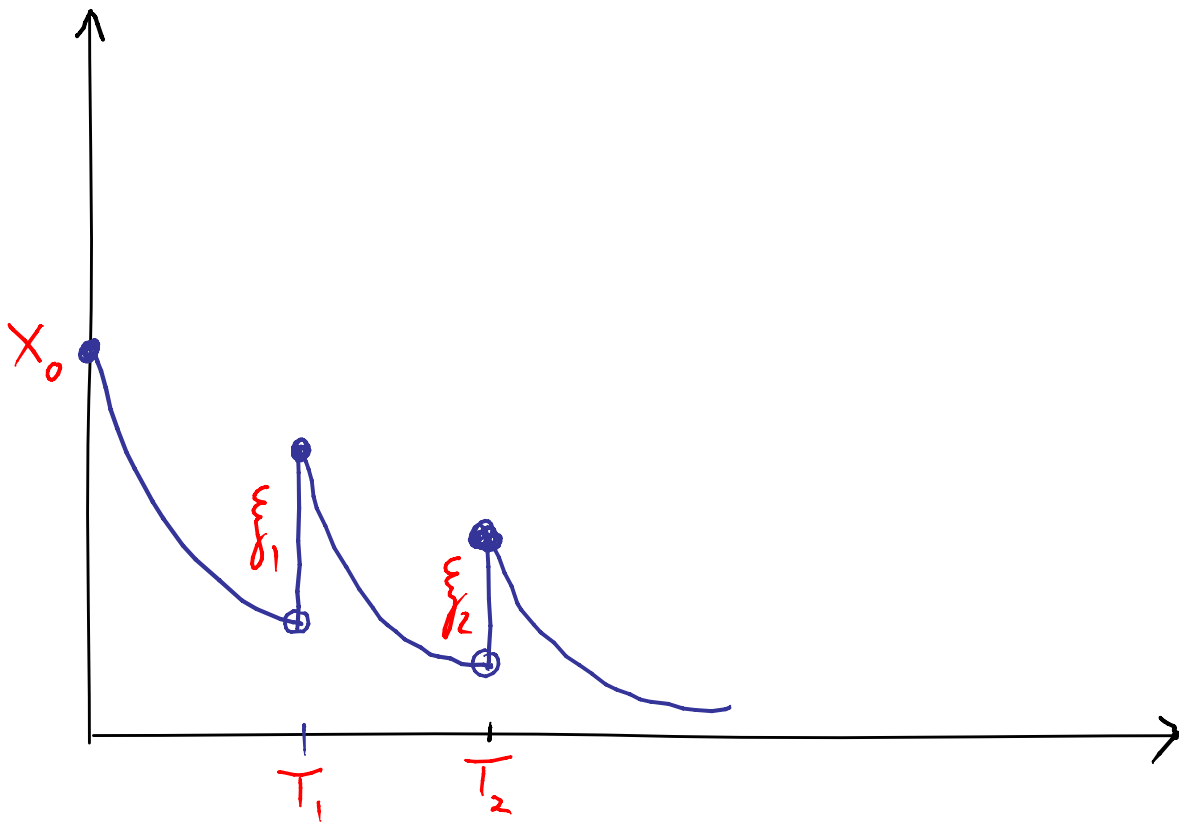$$c\mu_{\mathrm{gc}}(n)ET_1 = r\mu_{\mathrm{sn}}(n+1)ES_1 \ .$$

in particular

$$\mu_{\mathrm{sn}} = \frac{cET_1}{rES_1}$$

For càdlàg $Z = \{Z_t \mid t \geq 0\}$:

- $Z_{t-} = \lim\limits_{s \uparrow t} Z_s$

- $\Delta Z_t = Z_t - Z_{t-}$

- $\Delta Z_0 = Z_0$

- $Z_t^c = Z_t - \sum_{0 \leq s \leq t} \Delta Z_s$ when Z is BV

- $[Z, Z]_t$-quadratic variation
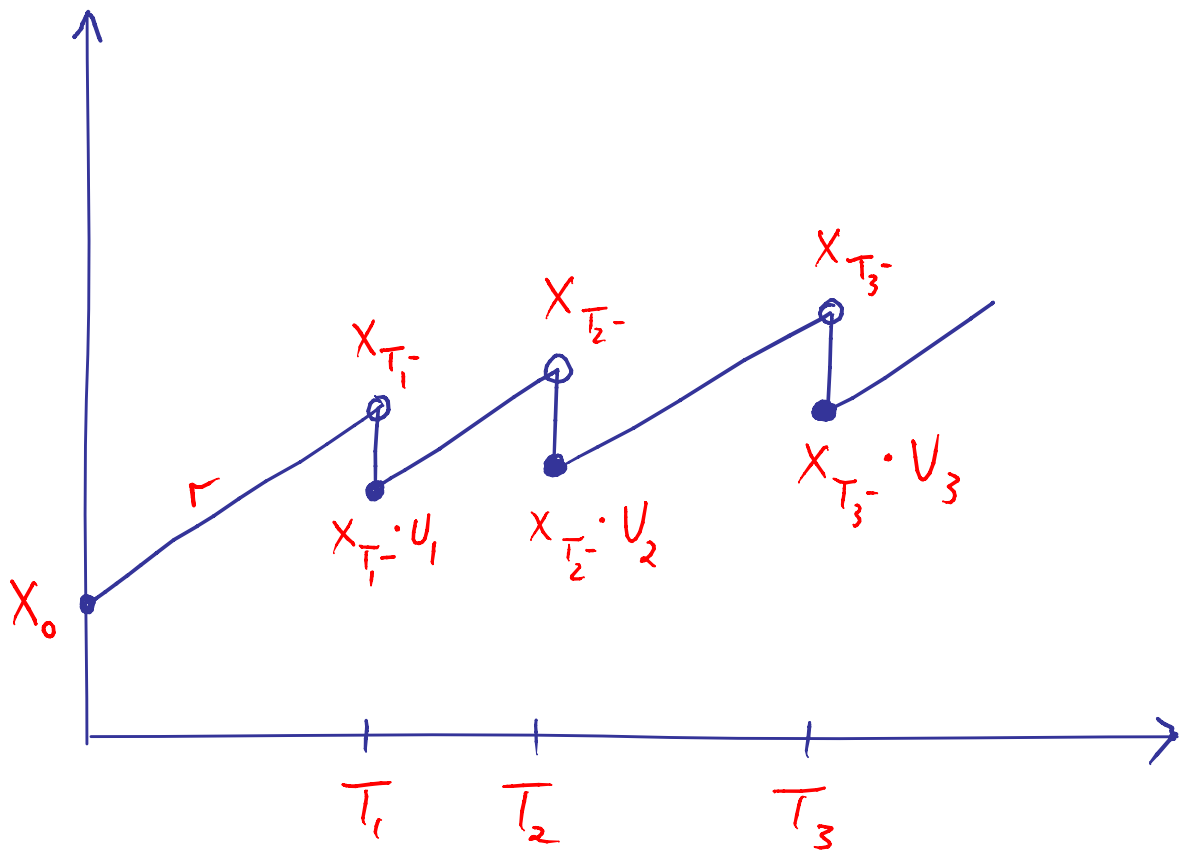
Shot-noise, linear dam

$$X_t = X_0 + Y_t - r \int_0^t X_s \, ds$$

$$Y_t = \sum_{i=1}^{N_t} \xi_i \qquad N_t = \sup\{n \mid T_n \le t\}$$

$$X_t = X_0 e^{-rt} + \sum_{i=1}^{N_t} \xi_i e^{-r(t-T_i)}$$

Growth - collapse
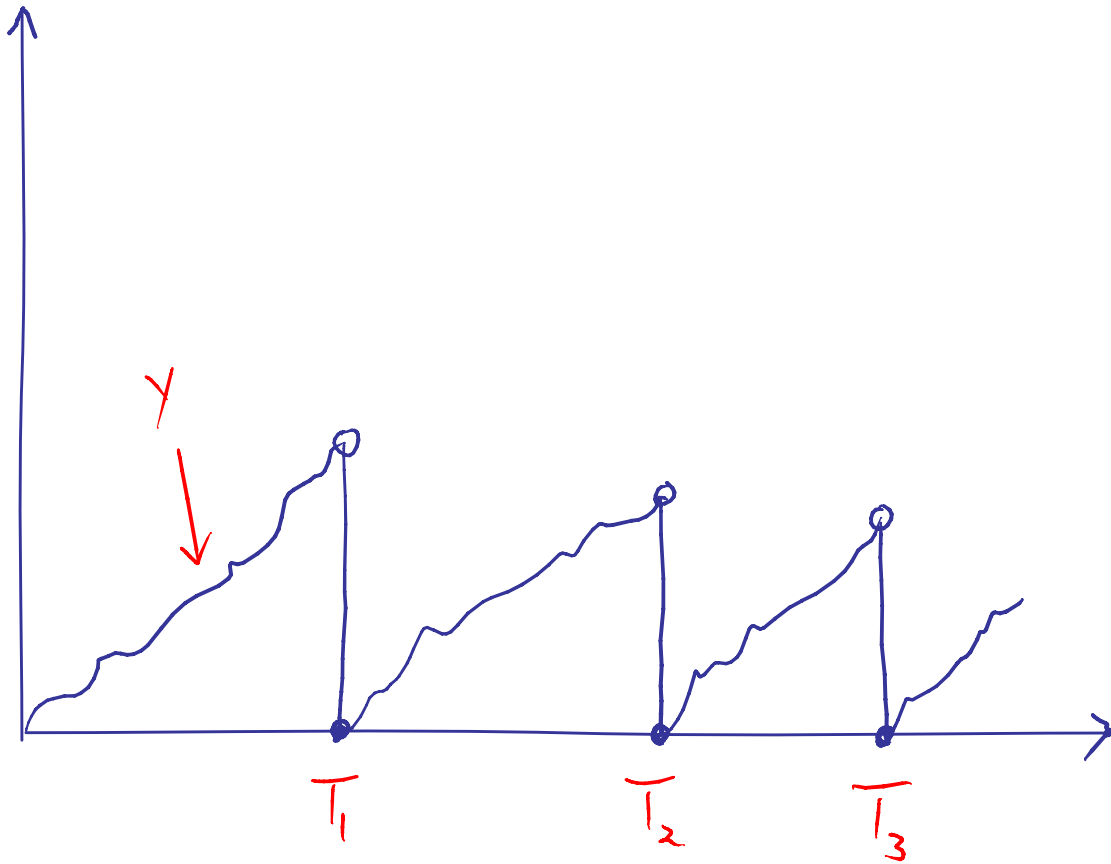

$$X_t = X_{T_{N_t}^-} \cdot U_{N_t} + r(t - T_{N_t})$$

$$X_{T_n^-} = X_{T_{n-1}^-} \cdot U_{n-1} + r(T_n - T_{n-1})$$

autoregressive (K. 2009)

$$X_t = X_0 + rt - \sum_{i=1}^{N_t} X_{T_i^-}(1 - U_i)$$

$$= X_0 + rt - \int_{(0,t]} X_{s^-} \, d \sum_{i=1}^{N_s}(1 - U_i)$$

Clearing process

$$X_t = Y_t - Y_{T_{N_t^-}}$$

$$X_{T_n^-} = Y_{T_n^-} - Y_{T_{n-1}^-}$$

$$X_t = Y_t - \sum_{i=1}^{N_t} \left(Y_{T_n^-} - Y_{T_{n-1}^-} \right) = Y_t - \int_{(0,t]} X_{s-} \, dN_s$$

$$\underbrace{\qquad\qquad\qquad}_{X_{T_n^-}}$$

Common structure: $\qquad$ Stoch. linear eq.

$$X_t = Y_t + \int_{(0,t]} X_{s-} \, dZ_s$$

Shot noise:

$$Y_t = X_0 + \sum_{i=1}^{N_t} \xi_i \, , \quad Z_t = -rt$$

Growth collapse:

$$Y_t = X_0 + rt \, , \quad Z_t = -\sum_{i=1}^{N_t} (1-U_i)$$

Clearing:

$$Y_t = \text{nondecreasing} \, , \quad Z_t = -N_t$$

$$\left\{ Y_{T_n+t} - Y_{T_n-} \mid t \geqslant 0 \right\} \sim Y$$

$$+ \text{ risk} + \text{finance}$$

Theorem 3 *Y, Z càdlàg, adapted, Y is BV, Z is a semimartingale. The unique càdlàg adapted solution to*

$$X_t = Y_t + \int_{(0,t]} X_{s-} \mathrm{d}Z_s$$

is

$$X_t = \int_{[0,t]} U_{u,t} \mathrm{d}Y_u$$

where $U_{t,t} = 1$ and for $u < t$, $U_{u,t} =$

$$e^{Z_t - Z_u - \frac{1}{2}([Z,Z]_t^c - [Z,Z]_u^c)} \cdot \prod_{u < s \leq t} (1 + \Delta Z_s) e^{-\Delta Z_s}$$

When Z is BV then for $u < t$, $U_{u,t} =$

$$e^{Z_t^c - Z_u^c} \prod_{u < s \leq t} (1 + \Delta Z_u)$$

Assume:

- Y, Z nondecreasing

- $Y_0 = Z_0 = 0$

- $\Delta Z_t \leq 1$

- Law of $(Y_{s+\cdot} - Y_s, Z_{s+\cdot} - Z_s)$ is independent of s (stationary increments)

- $X_t = X_0 + Y_t - \int_{(0,t]} X_{s-} \mathrm{d}Z_s$

- $N_t = \sum_{0 < s \leq t} 1_{\{\Delta Z_s = 1\}}$

- $T_n = \inf\{t|\ N_t = n\}$

- $J_t = Z_t^c - \sum_{0 < s \leq t} \log(1 - \Delta(Z_s - N_s))$
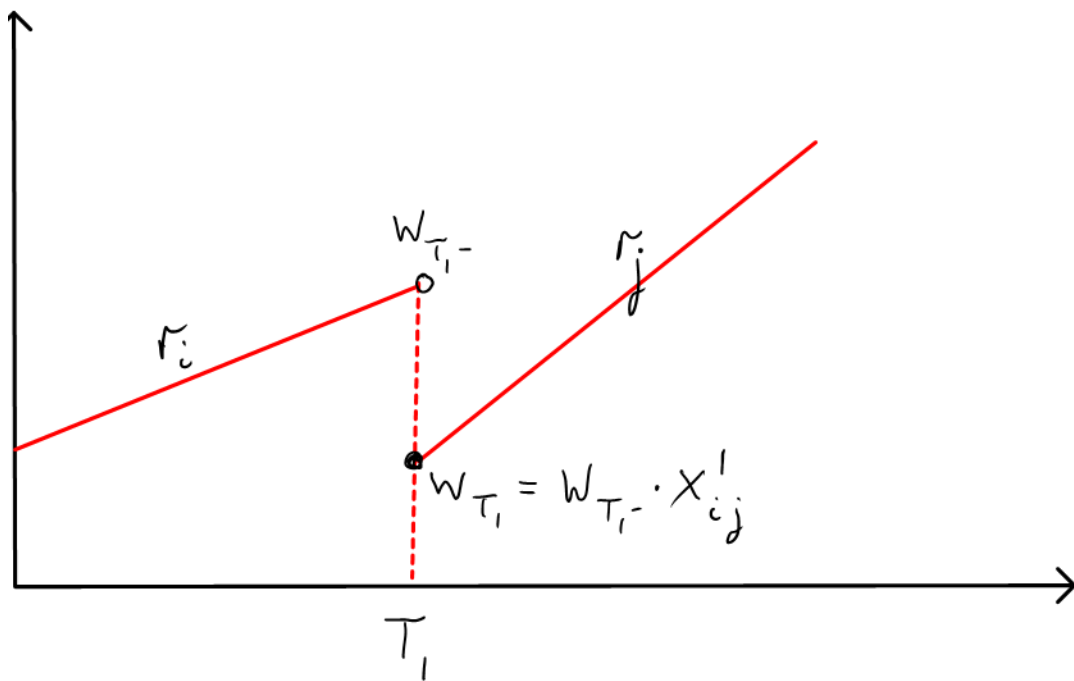
- Extend (Y, Z) to be a two sided process

Theorem 4 *If $\int_{(-\infty,0]} e^{J_s} \mathrm{d}Y_s < \infty$ a.s. and either $T_1 < \infty$ a.s. or $J_t \to \infty$ a.s. as $t \to \infty$, then X has the unique stationary version*

$$X_t^* = \int_{(-\infty,t]} e^{-(J_t - J_s)} 1_{\{N_t - N_s = 0\}} \mathrm{d}Y_s \qquad (8)$$

and for every initial a.s. finite X_0,

$$X_t \xrightarrow{\mathrm{d}} X_0^* ,$$

When $X_0 = 0$ a.s. then X_t is stochastically increasing in $t \geq 0$.

The figure shows two red line segments with a discontinuity. The upper segment is labeled r_i (left) and there is an open circle labeled W_{T_1-} at the top of the jump. Above the right segment is the label r_j. The filled dot at the bottom of the jump is labeled $W_{T_1} = W_{T_1-} \cdot X^1_{ij}$. A vertical dashed red line connects the two points at T_1 on the horizontal axis.

$$\{ J_t \mid t \geq 0 \} \quad CTMC, \ irred.$$

Rate trans. matrix $Q = (q_{ij})$

Stat. dist. $\pi = (\pi_i)$

$$d W_t = r_{J_t} \, dt \ - \ W_{t-} \cdot \left(1 - X^{N_t}_{J_{t-} J_t} \right) d N_t$$

$$N_t = \sup \{ n \mid T_n \leq t \}$$

$T_n -$ n^{th} state change epoch

$$X^n_{ij} - ind. \quad X^n_{ij} \sim X_{ij} \in [0,1]$$

States: $1, \ldots, K$.

Condition 1 *$\exists i, j$ such that $q_{ij} > 0$ and $P[X_{ij} = 1] < 1$.*

Theorem 1 *Under Condition 1 the process (W_t, J_t) has a well defined time stationary distribution which is also the limiting distribution, independent of initial conditions.*

(W_*, J_*) has the joint stationary distribution of $\{(W_t, J_t| \ t \geq 0\}$.

Extended generator:

$$\mathcal{A}f(x,i) = r_i f'(x,i)$$

$$+ \sum_{\substack{j=1 \\ j \neq i}}^{K} q_{ij}\left(Ef(xX_{ij},j) - f(x,i)\right)$$

$$= r_i f'(x,i) + \sum_{j=1}^{K} q_{ij} Ef(xX_{ij},j).$$

Theorem 2 $\forall a \geq 0$, $f(x, i) = c_i x^\alpha$ *is in the domain of* $\mathcal{A}$ *and thus, with* $a_{ij}(\alpha) = E X_{ij}^\alpha$,

$$\mathcal{A}f(x, i) = \alpha r_i c_i x^{\alpha-1} + x^\alpha \sum_{j=1}^{K} q_{ij} a_{ij}(\alpha) c_j$$

or

$$\mathcal{A}f(x) = \alpha x^{\alpha-1} D_r c + x^\alpha Q \circ A(\alpha) c,$$

where

$$
\begin{aligned}
A(\alpha) &= (a_{ij}(\alpha)) \\
A \circ B &= (a_{ij} b_{ij}) \\
D_r &= \text{diag}(r_1, \ldots, r_K) \\
c &= (c_i) \\
f(x) &= (f(x, i))
\end{aligned}
$$

and $\mathcal{A}$ *acts componentwise.*

$$\xi_i^n = E[W_*^n 1_{\{J^*=i\}}]$$
$$\xi^n = (\xi_i^n)$$

Lemma 1 *Let $D_q = diag(q_1, \ldots, q_K)$. Under Condition 1, the matrix $Q \circ A(\alpha)$ is nonsingular for every positive α and*

$$(-Q \circ A(\alpha))^{-1} \geq D_q^{-1} .$$

Theorem 3 *Under Condition 1,*

$$(\xi^n)^T = n! \pi^T \prod_{k=1}^{n} D_r(-Q \circ A(k))^{-1}$$

for $n \geq 1$.

Dynkin's martingale:

$$f(W_t, J_t) - f(W_0, J_0) - \int_0^t \mathcal{A}f(X_s, J_s)ds$$

Valid for $f(x, i) = c_i x^n$.
Set
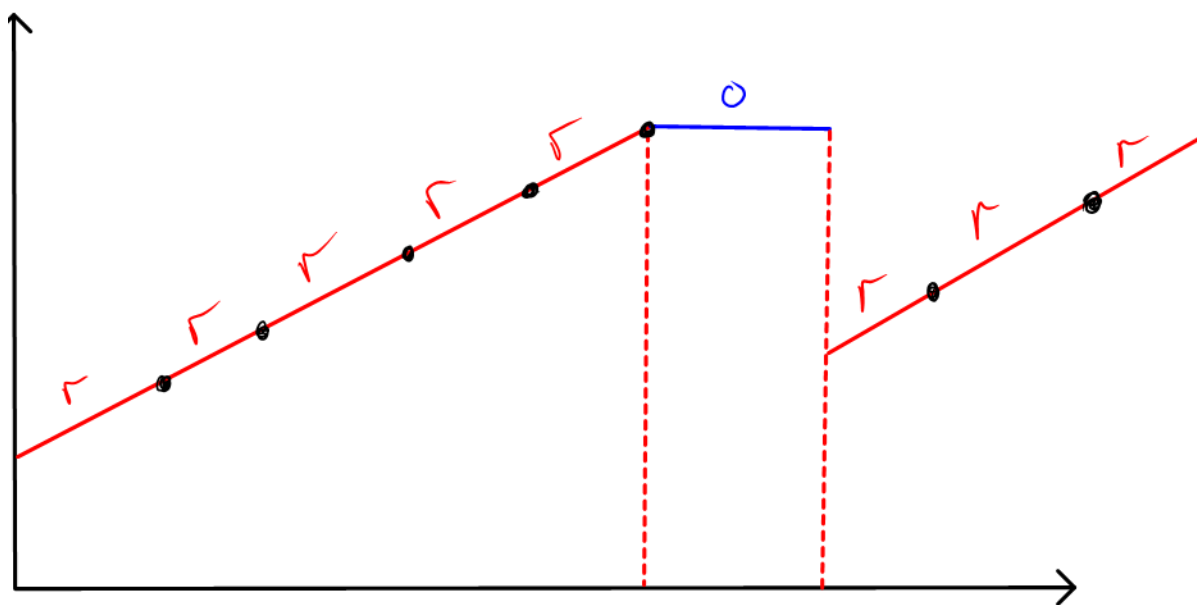$\xi_i^n(t) = E\left[W_t^n 1_{\{J_t=i\}}\right]$
$\xi^n(t) = (\xi_i^n(t))$
$c \in \mathbb{R}^K$
then

$$\begin{aligned}
\xi^n(t)^T c = \ & \xi^n(0)^T c \\
& + \int_0^t \left(n(\xi^{n-1}(s))^T D_r c \right. \\
& \left. + (\xi^n(s))^T Q \circ A(n)c\right) ds.
\end{aligned}$$

Thus

$$\frac{\mathrm{d}}{\mathrm{d}t}\xi^n(t) = nD_r\xi^{n-1}(t) + (Q \circ A(n))^T \xi^n(t)$$

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$$X_{i,j}^n = \begin{cases} X^n & j = 0 \\ 1 & j \neq 0 \end{cases}$$

$$X^n \sim i.i.d. \in [0,1]$$

Phase type inter-collapse times

States: $0, 1, \ldots, K$.

$$Q = \begin{pmatrix} -1 & \beta^T \\ -R\mathbf{1} & R \end{pmatrix}$$

$$P\left[\text{intercollapse time} > t\right] = \beta^T e^{Rt} \mathbf{1}$$

By regenerative theory,

$$F(t) = \frac{\sum_{i=1}^{K} P[W_* \leq t, J_* = i]}{1 - \pi_0}$$

$$\mu^n = \int_{[0,t]} t^n dF(t) = \frac{\sum_{i=1}^{K} \xi_i^n}{1 - \pi_0}$$

$a_{i0}(\alpha) = a(\alpha) = EX^\alpha$ for $i = 1, \ldots, K$ and $a_{ij}(\alpha) = 1$ for all other pairs.

$$Q \circ A(\alpha) = \begin{pmatrix} -1 & \beta^T \\ -a(\alpha)R\mathbf{1} & R \end{pmatrix}$$

Theorem 4 *For a growth collapse model with linear increase with rate $r > 0$, remaining proportion after a jump with distribution not concentrated at one, with nth moment $a(n)$ and with i.i.d. inter-collapse times having the phase type distribution $F(t) = 1 - \beta^T e^{Rt} \mathbf{1}$, a stationary distribution exists and has the following nth moment:*

$$\mu^n = n! r^n \frac{\beta^T (-R^{-1})}{\beta^T (-R^{-1}) \mathbf{1}}$$

$$\cdot \prod_{k=1}^{n} \left[\left(I + \frac{a(k)}{1 - a(k)} \mathbf{1} \beta^T \right) (-R^{-1}) \right] \mathbf{1} \, .$$

Corollary 1 *If in Theorem 4, in addition the remaining proportion after a jump is zero, then the growth collapse model becomes a clearing process and the corresponding moments are:*

$$\mu^n = n! r^n \frac{\beta^T (-R^{-1})^{n+1} \mathbf{1}}{\beta^T (-R^{-1}) \mathbf{1}} \ . \qquad (1)$$