*Andreas Löpker (joint work with Yoav Kerner)*

# ON THE STATIONARY EXCESS OPERATOR

Second Israeli-Dutch Workshop on
Queueing Theory

September 29-October 1, 2010

$$\mathcal{J}_\lambda^n F(x) = c_\lambda \int_0^\infty (1-F(x+u))\, u^{n-1} e^{-\lambda u}\, du$$

## Part I

## The Stationary Excess Operator

## Part II

## An insurance risk problem

## Part III

## The Overshoot Operator

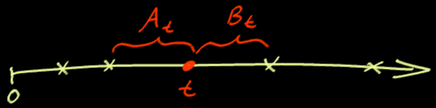
# Part I

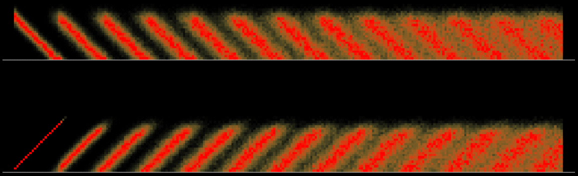
# The Stationary Excess Operator

## Renewal process

(i.i.d. interarrival times with distribution $F$)

- Remaining lifetime $B_t$
- Age $A_t$



$X \sim N(20, 2),\ t = 1, 2, \ldots, 250$

## Stationary excess operator

Suppose that $\mathbb{E}[X] < \infty$, then

$$\mathbb{P}(A_t \le x) \quad \to \quad \frac{1}{\mathbb{E}[X]} \int_0^x \big(1 - F(u)\big)\, du \quad \leftarrow \quad \mathbb{P}(B_t \le x)$$

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Define an Operator

$$\mathscr{S} F(x) = \frac{1}{\mathbb{E}[X]} \int_0^x \big(1 - F(u)\big)\, du$$

and its higher powers (provided $\mathbb{E}[X^n] < \infty$)

$$\mathscr{S}^n F(x) = \underbrace{\mathscr{S}\mathscr{S}\cdots\mathscr{S}}_{n\times} F(x).$$

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Interesting:

- Closed form for $\mathscr{S}^n F(x)$
- Moments, Laplace transform
- Convergence as $n \to \infty$

## Literature

HARKNESS & SHANTARAM (1969, 1972)

VAN BEEK & BRAAT (1973)
VARDI & SHEPP & LOGAN (1981)
WHITT (1985)
MASSEY & WHITT (1993)
LIN (1995)
HUNAG & LIN (1995)
PAKES (1996,2004)

## Properties of the stationary excess operator

Let $\mathscr{S}^n X$ be a random variable with distribution $\mathscr{S}^n F$.

- Closed form:

$$\mathscr{S}^n F(x) = 1 - \frac{\int_0^\infty (1 - F(u+x))\, u^{n-1}\, du}{\int_0^\infty (1 - F(u))\, u^{n-1}\, du}.$$

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- Laplace transform:

$$\frac{(-s)^n}{n!}\mathbb{E}[X^n] \cdot \mathbb{E}\left[e^{-s\mathscr{S}^n X}\right] = \varphi(s) - \sum_{k=0}^{n-1} \frac{(-s)^k}{k!}\mathbb{E}[X^k].$$

## Convergence as $n \to \infty$

$(\varrho(F) = \inf\{x : F(x) = 1\} < \infty)$ Then

$$n \cdot \mathscr{S}^n X \Rightarrow Z^1_{1/\varrho(F)},$$

where $Z^1_{1/\varrho(F)}$ is an exponential random variable with mean $\varrho(F)$.

## Convergence as $n \to \infty$

($\varrho(F) = \infty$) If $c = \limsup_{n\to\infty} \frac{c_n}{c_{n-1}} < \infty$ and

$$\mathscr{S}^n X / c_n \Rightarrow X^*,$$

then $F^*$ is continuous, concave and differentiable,

$$F^*(x) = \frac{1}{\mathbb{E}[X^*]} \int_0^{cx} (1 - F^*(u))\, du,$$

and $c = \lim_{n\to\infty} \frac{c_n}{c_{n-1}} \geq 1$ (if $c = 1$ then $F$ is exponential).

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In fact

$$1 - F^*(x) = \frac{\int_{-\infty}^{\infty} e^{-x/c^{u-1/2}} c^{-u^2/2}\, d\nu(u)}{\int_{-\infty}^{\infty} c^{-u^2/2}\, d\nu(u)},$$

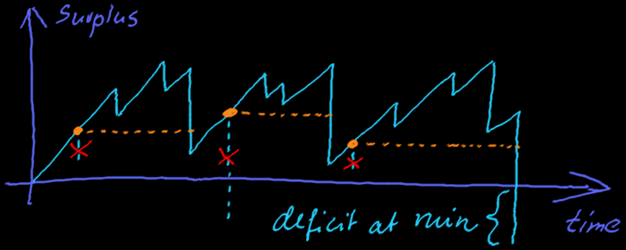
where $\nu(A+1) = \nu(A)$ and $\nu[0,1) = 1$.

# Part II

# An insurance risk problem

## Insurance problem

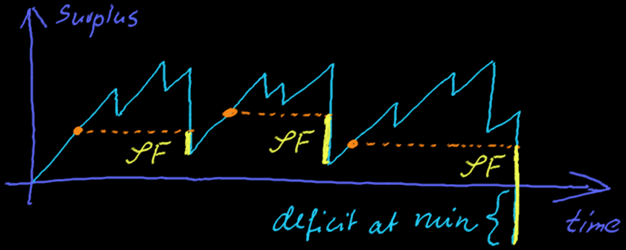
Risk model with Poisson arrival times, claim size distribution $F$.




- Reinsurance contract: First claim is ignored - surplus level $r$ is memorized.
- Whenever the surplus down-crosses $r$, the next claim is again ignored and the current level $r$ is memorized.

## Insurance problem

Risk model with Poisson arrival times, claim size distribution $F$.




For the Standard model: The conditional distribution of the deficit at ruin, given that ruin happens in finite time, is given by $\mathscr{S}F$.

**Insurance problem**

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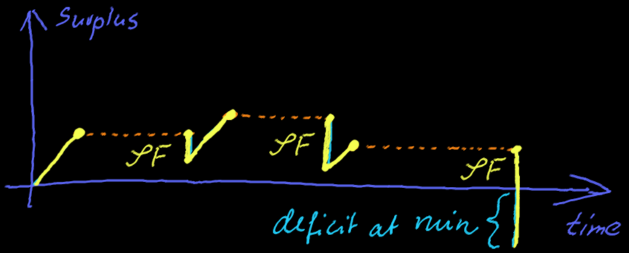


Remove the summits and glue the remaining parts together.

## Insurance problem

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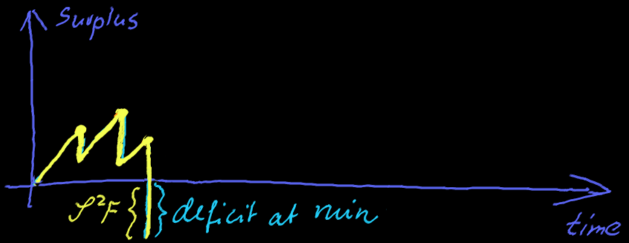
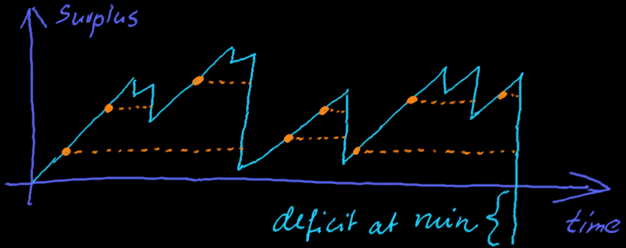



Standard model with claim size distribution $\mathscr{S} F$,

$\Rightarrow$ The deficit at ruin (conditioned to exist) has distribution $\mathscr{S}^2 F$.

## Insurance problem

Can be generalized (reinsurance covers first $n$ claims)

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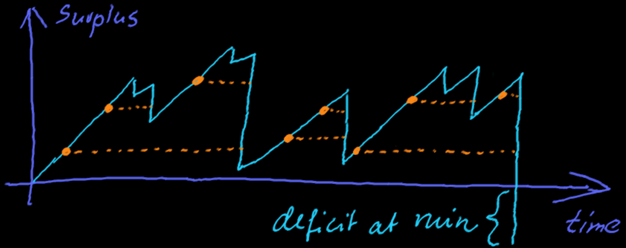



$\Rightarrow$ The deficit at ruin (conditioned to exist) has distribution $\mathscr{S}^n F$.
Net profit condition

$$\mathbb{E}[\mathscr{S}^k F] < \frac{1}{\lambda} \quad \Leftrightarrow \quad \frac{\mathbb{E}[X^{k+1}]}{\mathbb{E}[X^k]} < \frac{k+1}{\lambda}, \quad \forall k = 0, 1, \ldots, n$$

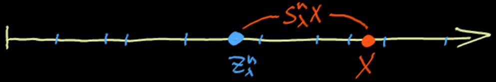
# Part III

# The Overshoot Operator

## New idea: $n^{th}$ overshoot operator

$Z_\lambda^n$ = Erlang-n variable with mean $n/\lambda$ and let

$$\mathscr{S}_\lambda^n F(x) = \mathbb{P}(X > Z_\lambda^n + x | Z_\lambda^n > X)$$



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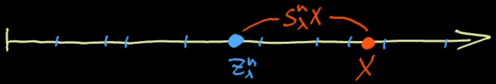


- $\mathscr{S}_\lambda^n$ is the $n^{th}$ power of $\mathscr{S}_\lambda^1$, i.e. $\mathscr{S}_\lambda^m \mathscr{S}_\lambda^n = \mathscr{S}_\lambda^{n+m}$

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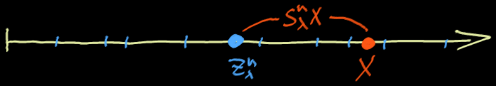


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- Explicit formula:

$$\mathscr{S}_\lambda^n F(x) = 1 - \frac{\int_0^\infty (1 - F(u+x))\, u^{n-1} e^{-\lambda u}\, du}{\int_0^\infty (1 - F(u))\, u^{n-1} e^{-\lambda u}\, du} = \frac{\mathrm{M}_{\lambda,n}(x)}{\mathrm{M}_{\lambda,n}}$$

where $\mathrm{M}_{\lambda,n}(x) = \int_0^\infty (1 - F(u+x))\, u^{n-1} e^{-\lambda u}\, du.$

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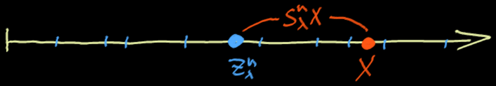
$$\mathscr{S}_\lambda^n F(x) = \mathbb{P}(X > Z_\lambda^n + x \mid Z_\lambda^n > X)$$



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where $\mathrm{M}_{\lambda,n}(x) = \int_0^\infty (1 - F(u+x))\, u^{n-1} e^{-\lambda u}\, du.$

- Approximation: $\mathscr{S}_\lambda^n F(x) \to \mathscr{S}^n F(x)$ as $\lambda \to 0$.

## Properties of the $n^{th}$ overshoot operator

- LST:

$$\frac{(\lambda - s)^n}{(n-1)!}\mathrm{M}_{\lambda,n} \cdot \mathbb{E}\left[e^{-s\mathscr{S}_\lambda^n X}\right] = \varphi(s) - \sum_{k=0}^{n-1} \frac{(s-\lambda)^k}{k!}\mathrm{D_k}\varphi(\lambda)$$

if $\lambda \neq s$ and

$$\varphi_{\lambda,n}(\lambda) = \frac{(-1)^n}{\mathrm{M}_{\lambda,n} n}\mathrm{D_n}\varphi(\lambda).$$

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- Moments:

$$\mathbb{E}\left[(\mathscr{S}_\lambda^n X)^m\right]\quad=\quad\frac{m!}{\mathrm{M}_{\lambda,n}}\sum_{k=0}^{\infty}\frac{(-\lambda)^k}{k!}\frac{(n+k-1)!}{(m+n+k-1)!}\mathrm{M}_{n+k+m}$$

## Convergence as $\lambda \to 0$

Suppose that $\mathbb{E}[X^{n+m}] < \infty$. Then

$$\mathscr{S}_\lambda^n F(x) = \mathscr{S}^n F(x) - \sum_{k=1}^{m} a_k (-\lambda)^k + o(\lambda^m),$$

as $\lambda \to 0$, where the coefficients $a_k$ can be evaluated via the recursive formula

$$\frac{m!}{\mathrm{M}_{n+m}} \sum_{i=0}^{m} \frac{\mathrm{M}_{n+m-i}}{(m-i)!} a_k = 1 - \mathscr{S}^{n+m} F(x).$$

## Convergence as $n \to \infty$

($\varrho(F) < \infty$) Then

$$n\mathscr{S}_\lambda^n X \overset{m}{\Rightarrow} Z^1_{1/\varrho(F)}.$$

## Convergence as $n \to \infty$

($\varrho(F) = \infty$) Suppose $c = \limsup_{n\to\infty} c_n / c_{n-1} < \infty$ and

$$\frac{\mathscr{S}_\lambda^n X}{c_n} \to X_\lambda^*. \tag{1}$$

Then $c = \lim_{n\to\infty} c_n / c_{n-1} \geq 1$.

1. If $c_n \to 0$ then

$$F_\lambda^*(x) = \frac{1}{\mathbb{E}[X_\lambda^*]} \int_0^{cx} (1 - F_\lambda^*(u))\, du. \tag{2}$$

   and $\mathbb{E}[X_\lambda^*] = \lim_{n\to\infty} \frac{n\mathrm{M}_{\lambda,n+1}}{c_n \mathrm{M}_{\lambda,n}}$.

2. If $c_n \to q$ for some $q \in (0,\infty)$ then $\mathscr{S}_\lambda^n X / c_n \to Z_{\beta(\lambda)}^1$ where $Z_{\beta(\lambda)}^1$ has exponential distribution with rate

$$\beta(\lambda) = q \times \left( \lim_{n\to\infty} n \frac{\mathrm{M}_{\lambda,n}}{\mathrm{M}_{\lambda,n+1}} - \lambda \right).$$

## A useful observation

Define the operator

$$\mathscr{T}_\lambda F(x) = \mathbb{P}\left(X \wedge Z_\lambda^1 \le x\right) = 1 - e^{-\lambda x}(1 - F(x)).$$

Then

$$\mathscr{T}_\lambda \mathscr{S}_\lambda^n = \mathscr{S}^n \mathscr{T}_\lambda$$

(Formally $\mathscr{S}_\lambda^n = \mathscr{T}_{-\lambda} \mathscr{S}^n \mathscr{T}_\lambda$)

# Thank you!