Scaling Limits,
Cyclically Varying Birth-Death Processes
and Stationary Distributions

Matthieu Jonckheere [1],
Yoni Nazarathy [2].

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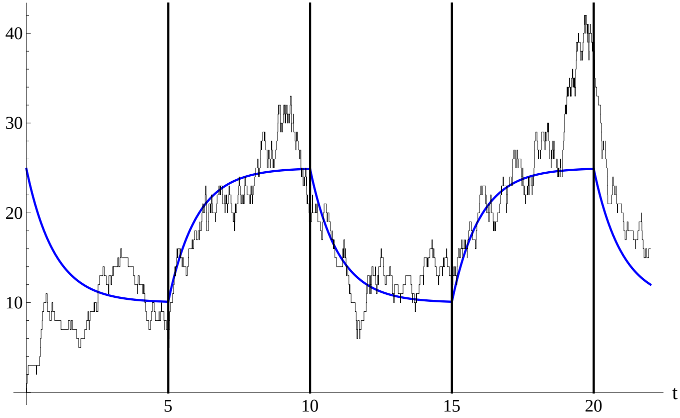
[1] TU/e and CONICET, University of Buenos Aires

[2] EURANDOM and TU/e, supported by NWO-VIDI grant 639.072.072 of Erjen Lefeber

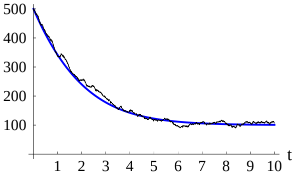
Overview

- Birth death processes
- Scaling limits
- Cyclically varying systems
- Stationary distributions

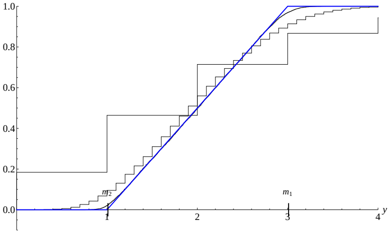


Talk Outline

- Part 1: Approximating trajectories (of birth-death processes)



- Part 2: Approximating stationary distributions (of cyclic processes)



Part 1: Approximating Trajectories

An Example Class of Birth Death Processes

- $\{X(t), t \geq 0\}$ is a Continuous Time, Birth-Death, Markov Chain taking values $\{0, 1, \ldots\}$
- Birth rates are constant: λ
- Death rates are state dependent: $\mu X(t)^{\alpha}$, $\alpha \geq 0$
- $\alpha = 0$ is M/M/1, $\alpha = 1$ is M/M/∞

Desired: A deterministic $x(t)$ that approximates $X(t)$

Some ideas: R.W.R. Darling, J.R. Norris, *Differential equation approximations for Markov chains*, Probability Surveys, 5, pp. 37-79, 2008

Scaling The Processes

A sequence of processes

- $X_N(\cdot)$, $N = 1, 2, \ldots$
- The parameters of the N'th process: λ_N, μ_N and α
- Initial values are $X_N(0) = N\,X(0)$
- Desired: $X_N(t) \approx N\,x(t)$ as $N \to \infty$ (for finite t)

Try $x(t)$, solution of the ODE:

$$
\begin{aligned}
\dot{x}(t) &= \lambda - \mu x(t)^\alpha \\
x(0) &= X(0)
\end{aligned}
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What is a "correct" scaling?

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What is a "correct" scaling?

Observe from the ODE:

$$\lambda_N = \lambda N, \mu_N = \mu N^{1-\alpha}$$

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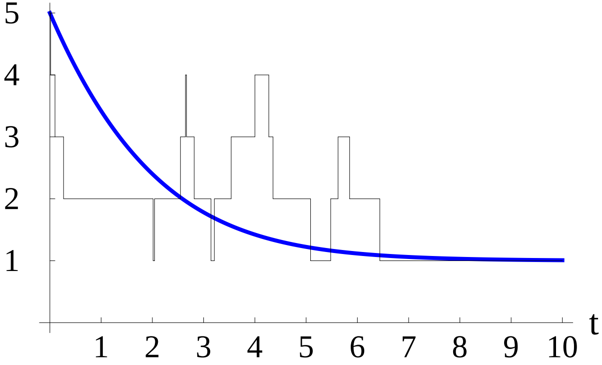
Illustration for $\alpha = 2/3$

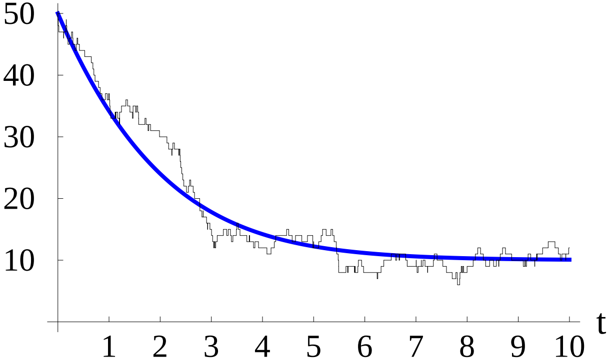
$$N = 1$$



$$\lambda = \mu = 1, \quad X(0) = 5$$

Illustration for $\alpha = 2/3$

$N = 10$



$\lambda = \mu = 1, \quad X(0) = 5$

[redacted]

Illustration for $\alpha = 2/3$

$$N = 100$$

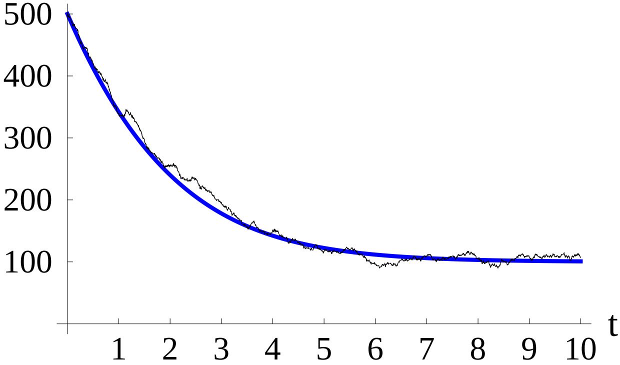


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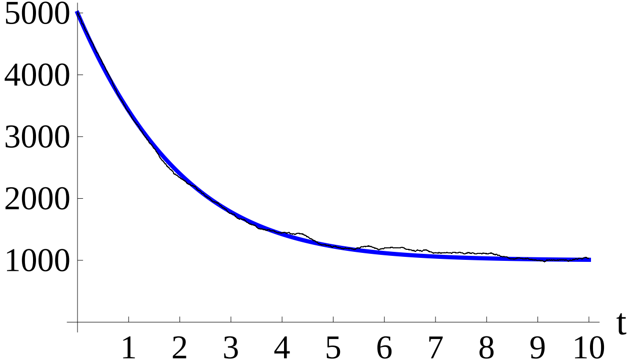
Illustration for $\alpha = 2/3$

$$N = 1000$$



$$\lambda = \mu = 1, \quad X(0) = 5$$

Theorem

(i) Trajectories:

$$\lim_{N\to\infty} P\Big(\sup_{s\in[0,t]}\Big|\frac{X_N(s)}{N}-x(s)\Big|>\epsilon\Big)=0$$

(ii) Hitting Times:

$$\lim_{N\to\infty} P\Big(\Big|\mathcal{T}_N(yN)-\tau(y)\Big|>\epsilon\Big)=0$$

where,

$$\mathcal{T}_N(y)=\inf\{t:X_N(t)=y\},\quad \tau(y)=\inf\{t:x(t)=y\}=x^{-1}(y)$$

Note 1: For $\alpha=0,1$ it is well known, see P. Robert book, 2003
Note 2: Also have formulation for more general BD processes

Martingale Representation

$$X_N(t) = X_N(0) + M_N(t) + \lambda_N t - \mu_N \int_0^t X_N(s)^\alpha ds$$

Substitute: $X_N(0) = N\, X(0)$, $\lambda_N = \lambda N$, $\mu_N = \mu N^{1-\alpha}$ and divide by N

$$\frac{X_N(t)}{N} = X(0) + \frac{M_N(t)}{N} + \lambda t - \mu \int_0^t \left(\frac{X_N(s)}{N}\right)^\alpha ds$$

Compare With the Deterministic Trajectory:

$$x(t) = X(0) + \lambda t - \mu \int_0^t x(s)^\alpha ds$$

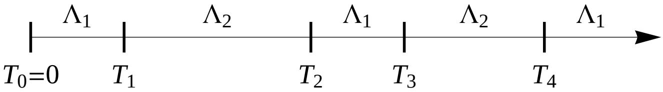
$$sup_{s\in[0,t]}\left|\frac{X_N(s)}{N} - x(s)\right| \leq sup_{s\in[0,t]}\frac{|M_N(s)|}{N} + \int_0^t sup_{u\in[0,s]}\left|\left(\frac{X_N(u)}{N}\right)^\alpha - x(u)^\alpha\right| ds$$

[redacted]

Part 2: Approximating Stationary Distributions
(of Cyclically Varying Systems)

Cyclically Varying Systems

- A sequence of increasing time points $\{T_n, n \geq 0\}$
- Two sets of birth-death parameters $\Lambda_i = \big(\lambda_i, \mu_i\big)$, $i = 1, 2$
- At time points T_n, $X(t)$ changes behavior, alternating between Λ_1 and Λ_2



Types of Cyclic Behavior

Hysteresis Control

$$T_n = \inf\left\{t > T_{n-1} : X(t) = \left\{\begin{array}{ll} \ell_2 & n \text{ odd} \\ \ell_1 & n \text{ even} \end{array}\right. \right\}$$

Fixed Cycles

$$T_n - T_{n-1} = \left\{\begin{array}{ll} \tau_1 & n \text{ odd} \\ \tau_2 & n \text{ even} \end{array}\right.$$

Random Environment

$$T_n - T_{n-1} \sim \left\{\begin{array}{ll} \exp\left(\tau_1^{-1}\right) & n \text{ odd} \\ \exp\left(\tau_2^{-1}\right) & n \text{ even} \end{array}\right.$$

Some of the Related Literature

Hysteresis Control

Federgruen and Tijms 1980, Perry 1997, Bekker 2009...

Fixed Cycles

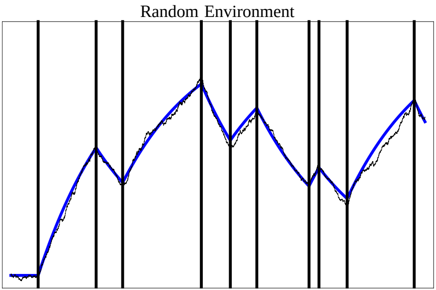
Harrison and Lemoine 1977, Lemoine 1989, Breuer 2004...

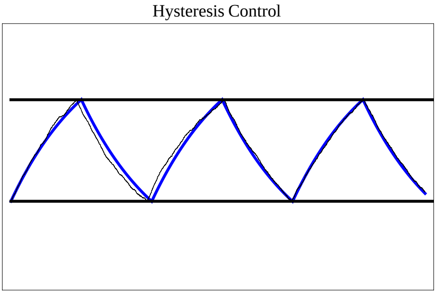
Random Environment

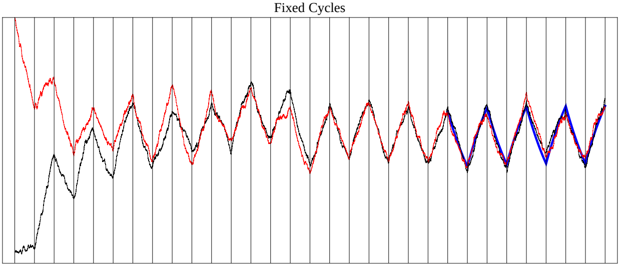
Yechiali and Naor 1971, Neuts 1977, Prabhu and Zhu 1989, Boxma and Kurkova 2000, Falin 2008, Fralix and Adan 2009...

In general, the queue level distribution is "tough". Things get "tougher" as one moves from $\alpha = 0$ to $\alpha = 1$ and then to arbitrary α.

Basic Idea: Use the Scaling Limits







Basic Idea: Use the Scaling Limits

Hysteresis Control

Look at one deterministic cycle through $\ell_1 \to \ell_2 \to \ell_1$

Fixed Cycles

Look at one deterministic cycle of duration $\tau_1 + \tau_2$

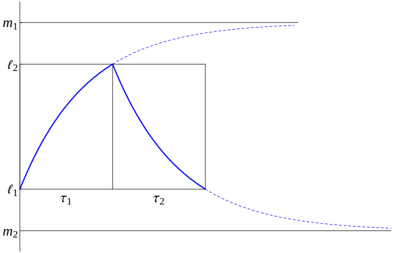
Random Environment

Look at a piece-wise deterministic Markov process (PDMP)

In all 3 cases: Construct a distribution function $F(\cdot)$
by means of the scaling limit

$F(\cdot)$ for Hysteresis Control and Fixed Cycles

$\dot{x}_i(t) = \lambda_i - \mu_i x(t)^\alpha$
$x_i(0) = \ell_i$
$\lim_{t\to\infty} x_i(t) = m_i$
$m_2 < \ell_1 < \ell_2 < m_1$
$\dot{x}_2(0) < 0 < \dot{x}_1(0)$
$\tau_i(y) = \inf\{t : x_i(t) = y\}$
$\tau_i = \tau_i(\ell_{\bar{i}})$



A CDF with support $[\ell_1, \ell_2]$, (assume $\alpha > 0$)

$$F(y) = \frac{1}{\tau_1 + \tau_2}\big(\tau_1(y) + (\tau_2 - \tau_2(y))\big)$$

- For Hysteresis control, ℓ_1, ℓ_2 given, τ_1, τ_2 easily calculated
- For Fixed Cycles τ_1, τ_2 given, unique ℓ_1, ℓ_2 obtained by solving:

$$x_1\Big|_{x_1(0)=\ell_1}^{(\tau_1)} = \ell_2, \qquad x_2\Big|_{x_2(0)=\ell_2}^{(\tau_2)} = \ell_1$$

$F(\cdot)$ for Random Environment

- PDMP: Environment Markov chain alternates between $1, 2$. Given a mode, trajectory is deterministic with "state-dependent" rates.
- O. Kella and W. Stadje, *Exact Results for a Fluid Model with State-Dependent Flow Rates*, Prob. in Eng. and Inform. Sci., 16, pp. 389-402, 2002.

Stationary Distribution

Solve for $p_1(\cdot), p_2(\cdot)$ on $y \in (m_2, m_1)$

$$
\begin{aligned}
(\lambda_1 - \mu_1 y^\alpha) p_1'(y) &= {\tau_2}^{-1} p_2(y) - {\tau_1}^{-1} p_1(y) \\
(\lambda_2 - \mu_2 y^\alpha) p_2'(y) &= {\tau_1}^{-1} p_1(y) - {\tau_2}^{-1} p_2(y) \\
p_1(m_2) = 0, &\qquad p_2(m_1) = \frac{\tau_2}{\tau_1 + \tau_2}
\end{aligned}
$$

$$F(y) = p_1(y) + p_2(y), \quad y \in (m_2, m_1)$$

Some Cases where $F(\cdot)$ is explicit

Hysteresis Control or Fixed Cycles where $\alpha = 1$

$$F(y) = \int_{-\infty}^{y} f(u)du, \qquad f(u) = \frac{\frac{(\mu_1-\mu_2)u+(\lambda_2-\lambda_1)}{(\mu_1 u-\lambda_1)(\mu_2 u-\lambda_2)}}{\log\left(\frac{\mu_1\ell_1-\lambda_1}{\mu_1\ell_2-\lambda_1}\right)^{\frac{1}{\mu_1}}\left(\frac{\mu_2\ell_2-\lambda_2}{\mu_2\ell_1-\lambda_2}\right)^{\frac{1}{\mu_2}}}\mathbf{1}_{\{\ell_1\le u\le\ell_2\}}$$

For fixed cycles set: $\ell_i = \frac{(e^{\tau_i\mu_i}-1)\frac{\lambda_i}{\mu_i}+(e^{\tau_{\bar{i}}\mu_{\bar{i}}}-1)\frac{\lambda_{\bar{i}}}{\mu_{\bar{i}}}e^{\tau_i\mu_i}}{e^{\tau_i\mu_i+\tau_{\bar{i}}\mu_{\bar{i}}}-1}$

Hysteresis Control or Fixed Cycles with $\alpha = 0$

Uniform distribution, sometimes with masses at the endpoints

Random Environment with $\alpha = 0$

Truncated exponential distribution with masses at m_1 and m_2

Random Environment with $\alpha = 1$

When $\mu_1 = \mu_2 = \tau_1 = \tau_2 = 1$, uniform on $[\lambda_2, \lambda_1]$. Otherwise, more complex explicit expression

Convergence of Stationary Distributions

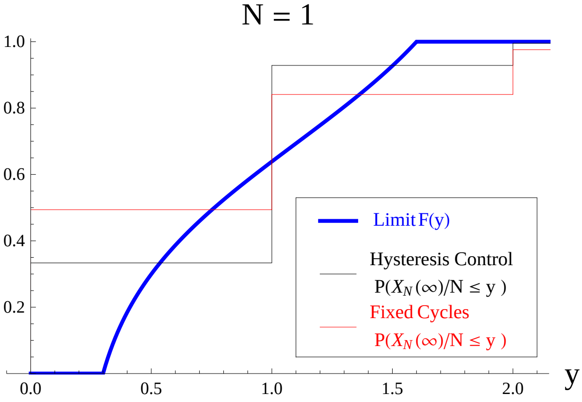
Assume $X_N(\cdot)$ is positive-recurrent. Then,

$$\lim_{N\to\infty} \sup_{y} \left| P\big(\frac{X_N(\infty)}{N} \leq y\big) - F(y) \right| = 0,$$

In the hysteresis control case, also scale the thresholds: $\Big(\lceil N\ell_1 \rceil, \lfloor N\ell_2 \rfloor\Big)$

Numerical Example: Hysteresis Control and Fixed Cycles

$$\begin{array}{cccc} \alpha = 1 & \mu_1 = \mu_2 = 1 & \lambda_1 = 2 & \lambda_2 = 0.2 \\ \ell_1 = 0.3 & \ell_2 = 1.6 & \tau_1 = 1.447 & \tau_2 = 2.639 \end{array}$$



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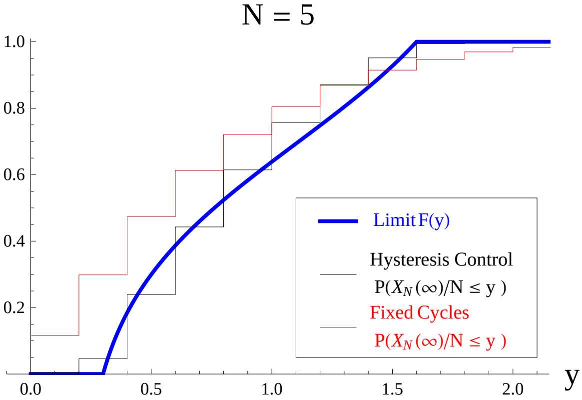

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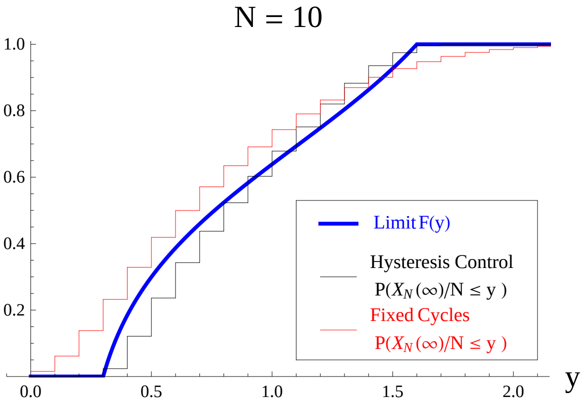

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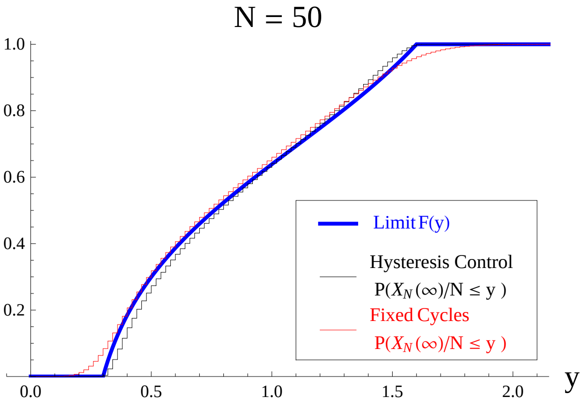

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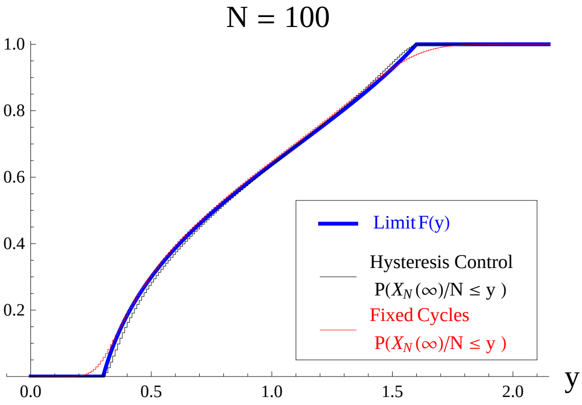

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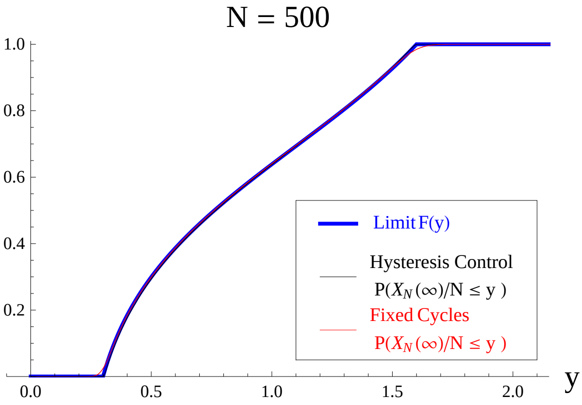
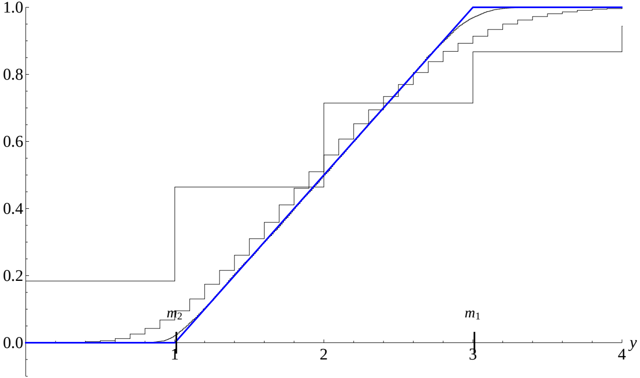


Numerical Example: Random Environment - Uniform

$$\alpha = 1$$

$$\mu_1 = \mu_2 = \tau_1 = \tau_2 = 1,\ \lambda_1 = 3,\ \lambda_2 = 1,$$

$$N = 1, 10, 100:$$

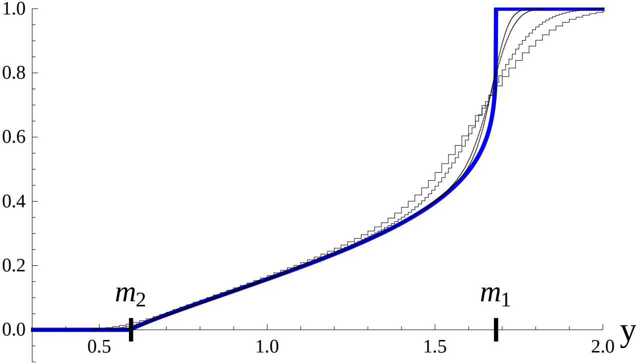


Numerical Example: Random Environment

$$\alpha = 4/3$$

$$\mu_1 = \mu_2 = 1,\ \ \lambda_1 = 2,\ \ \lambda_2 = 1/2,\ \ \tau_1 = 3,\ \ \tau_2 = 1$$

$$N = 50, 100, 500, 1000:$$



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Questions?