Two Queues Driven by a Birth-Death Process and Coupled by Overflow, Jockeying, and Customers Acting as Servers

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General setting

Two interacting queues, Q_1 and Q_2, with the following special features:
(i) regulated overflow from the first to the second queue;
(ii) state-dependent arrival and service rates;
(iii) the customers in the first queue act as servers for the second one;
(iv) jockeyeing mechanism from the second queue to the first one.

The basic model

- number of customers in Q_1: a birth and death chain with state space $\{0, \ldots, N\}$
- birth rates $\lambda_{1,n} > 0$, $n \in \{0, \ldots, N-1\}$, death rates $\mu_{1,n} > 0$, $n \in \{1, \ldots, N\}$, and $\mu_{1,0} = 0$
- $\lambda_{1,N} > 0$ arrival rate if Q_1 is in state N.
- Q_2: infinite waiting room, service rate $n\mu_2$ as long as n customers are present in Q_1
- If Q_1 is fully occupied, a new arrival of Q_1 moves to Q_2 with probability p and leaves with probability $1-p$.

Extensions

- Q_2 may have its own Poisson arrival stream, independent of Q_1, so that its arrival rate is λ_2 as long as Q_1 is not fully occupied and $p\lambda_{1,N} + \lambda_2$ if it is
- *Limited jockeying*: k customers $(1 \leq k \leq N)$ of Q_2, provided k or more are present, are forced to move to Q_1 as soon as Q_1 empties.
- *Exhaustive jockeying*: as soon as Q_1 is not fully occupied, it is filled with customers from Q_2 until it reaches its capacity bound or Q_2 empties.

Generalization of Perel and Yechiali (2008): special case $\lambda_{1,n} = \lambda_1 > 0$, $\mu_{1,n+1} = \mu_1 > 0$ for $n = 0, \ldots, N-1$, $\lambda_2 > 0$, $p = 0$ (no overflow), no jockeying.

The basic model: steady-state equations

Aim: to determine the distribution of the two-dimensional queue lengths process for the various models
$\longrightarrow$ irreducible Markov chain with state space

$$\{(n,m) \mid n = 0, \dots, N,\ m \in \mathbb{Z}_+\}$$

$(n,m) \ = \$ numbers of customers present in Q_1 and Q_2, resp.

$(L_1, L_2) \ = \$ steady-state queue lengths of (Q_1, Q_2)

The first queue

The first queue is a one-dimensional birth and death chain with finite state space; thus the steady-state distribution of L_1 exists and is well-known:

$$\mathbb{P}(L_1 = n) = \frac{\prod_{i=0}^{n-1} \rho_i}{1 + \sum_{j=1}^{N} \prod_{i=0}^{j-1} \rho_i} \quad \text{and} \quad \mathbb{E}L_1 = \frac{\sum_{n=1}^{N} n \prod_{i=0}^{n-1} \rho_i}{1 + \sum_{n=1}^{N} \prod_{i=0}^{n-1} \rho_i}, \qquad (1)$$

where $\rho_n = \lambda_{1,n}/\mu_{1,n+1}$, $n = 0, \ldots, N-1$.

Examples: Q_1

With constant arrival and service rates $\lambda_{1,n} = \lambda_1$, $\mu_{1,n} = \mu_1$ we get the standard $M/M/1/N-1$ queue: For $\lambda_1 \neq \mu_1$,

$$\mathbb{P}(L_1 = n) = \frac{\left(\frac{\lambda_1}{\mu_1}\right)^n - \left(\frac{\lambda_1}{\mu_1}\right)^{n+1}}{1 - \left(\frac{\lambda_1}{\mu_1}\right)^{N+1}} \text{ and } \mathbb{E}L_1 = \frac{\lambda_1}{\mu_1 - \lambda_1} - \frac{(N+1)\lambda_1^{N+1}}{\mu_1^{N+1} - \lambda_1^{N+1}}.$$

In the case $\lambda_1 = \mu_1$, we have $\mathbb{P}(L_1 = n) = 1/(N+1)$ and $\mathbb{E}L_1 = N/2$. For service rates $\mu_{1,n} = n\mu_1$ and constant arrival rate $\lambda_{1,n} = \lambda_1$ the first queue is a classical $M/M/N/0$ Erlang loss system.

Steady-state probabilities

$$p_{n,m} \equiv \mathbb{P}(L_1 = n, L_2 = m), \quad n = 0, \dots N \text{ and } m \in \mathbb{Z}_+$$

In the stable case:
$(p_{n,m})_{0 \le n \le N,\, m \ge 0}$ is the unique nonnegative and normalized solution of the steady-state equations:

$$\begin{aligned}
&(\lambda_{1,n}(1-\delta_{nN}) + p\lambda_{1,N}\delta_{nN} + (1-\delta_{n0})\mu_{1,n} + (1-\delta_{m0})n\mu_2)p_{n,m} \\
&\quad = (1-\delta_{n0})\lambda_{1,n-1}p_{n-1,m} + (1-\delta_{nN})\mu_{1,n+1}p_{n+1,m} \\
&\qquad + (1-\delta_{m0})\delta_{nN}p\lambda_{1,N}p_{n,m-1} + (1-\delta_{n0}(1-\delta_{m0}))n\mu_2 p_{n,m+1}
\end{aligned} \tag{2}$$

for $n = 0, \dots, N$ and $m \ge 0$ (where $\delta_{ij} = 1$ for $i = j$ and 0 otherwise).

Stability condition

First conjecture: $p\lambda_{1,N} < \mu_2 \mathbb{E} L_1$. However, on second thought, probably:

$$p\lambda_{1,N}\mathbb{P}(L_1 = N) < \mu_2 \mathbb{E} L_1. \tag{3}$$

Proposition

The system (2) has a unique nonnegative and normalized solution if and only if (3) holds, with $\mathbb{P}(L_1 = N)$ and $\mathbb{E}L_1$ as given above, or, equivalently,

$$p\lambda_{1,N} \prod_{n=0}^{N-1} \rho_n < \mu_2 \sum_{n=1}^{N} n \prod_{i=0}^{n-1} \rho_i$$

holds, where $\rho_n = \lambda_{1,n}/\mu_{1,n+1}$ for $n = 0, \ldots, N-1$.

Proof. Theory of quasi birth and death processes, regarding Q_1 as the phase process and Q_2 as the level process.

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Generating functions and steady-state distribution

Method of Avi-Itzhak and Mitrani (1968) and Perel and Yechiali (2008): probability generating function of the queue length of Q_2 for fixed queue length n of Q_1:

$$G_n(z) = \sum_{m=0}^{\infty} p_{n,m} z^m, \quad |z| \leq 1, \quad n = 0, \dots, N.$$

From the steady-state equations, for $n = 1, \dots, N-1$:

$$\begin{aligned} \lambda_{1,0} G_0(z) &= \mu_{1,1} G_1(z), \\ \big((\lambda_{1,n} + \mu_{1,n})z - n\mu_2(1-z)\big) G_n(z) &= \lambda_{1,n-1} z G_{n-1}(z) + \mu_{1,n+1} z G_{n+1}(z) \\ &\quad - n\mu_2(1-z) p_{n,0} \end{aligned}$$

$$\begin{aligned} &\big(\mu_{1,N} z + (p\lambda_{1,N} z - N\mu_2)(1-z)\big) G_N(z) \\ &\qquad = \lambda_{1,N-1} z G_{N-1}(z) - N\mu_2(1-z) p_{N,0}. \end{aligned}$$

Equations in matrix form

Define the $((N+1)\times(N+1))$-matrix $A(z)$ by

$$A(z) = \begin{pmatrix} \alpha_0(z) & -\mu_{1,1} & 0 & \dots & \dots & 0 \\ -\lambda_{1,0}z & \alpha_1(z) & -\mu_{1,2}z & \ddots & & \vdots \\ 0 & \ddots & \alpha_2(z) & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & & \ddots & \ddots & \ddots & -\mu_{1,N}z \\ 0 & \dots & \dots & 0 & -\lambda_{1,N-1}z & \alpha_N(z) \end{pmatrix},$$

where

$$\begin{aligned} \alpha_0(z) &= \lambda_{1,0}, \\ \alpha_n(z) &= (\lambda_{1,n}+\mu_{1,n})z - n\mu_2(1-z) \text{ for } n = 1,\dots,N-1 \text{ and} \\ \alpha_N(z) &= \mu_{1,N}z + (p\lambda_{1,N}z - N\mu_2)(1-z). \end{aligned}$$

Equations in matrix form (continuation)

Define

$$G(z) = (G_0(z), \dots, G_N(z))^\top,$$
$$P = (0, p_{1,0}, 2p_{2,0}, \dots, Np_{N,0})^\top.$$

$$\implies \quad A(z)G(z) = -\mu_2(1-z)P.$$

Let $A_n(z)$ be the matrix obtained from $A(z)$ by replacing the $(n+1)$th column by the vector $-\mu_2(1-z)P$ for $n = 0, \dots, N$. By Cramer's rule we can write

$$\det(A(z))G_n(z) = \det(A_n(z)) \tag{4}$$

The generating functions $G_0, \dots, G_N$ are uniquely determined by the equations (4) and thus by $p_{1,0}, \dots, p_{N,0}$, since these are the only unknowns occurring in the equations.

The roots of $\det(A(z))$

Theorem

$\det(A(z))$ *is a polynomial of degree* $N+1$ *and has* $N-1$ *distinct zeros in the interval* $(0,1)$ *and one zero at* $z=1$*. Moreover,* $\det(A(z))$ *has another zero in the interval* $(1,\infty)$ *if and only if the queueing system is stable.*

Finding $p_{1,0}, \ldots, p_{N,0}$

We can use the $N-1$ zeros $z_1, \ldots, z_{N-1}$ of $\det(A(z))$ in $(0,1)$ to find the N unknown probabilities $p_{1,0}, \ldots, p_{N,0}$:

$$\det(A_0(z_1)) = 0, \ldots, \det(A_0(z_{N-1})) = 0.$$

One more equation relating the unknowns is

$$\sum_{n=1}^{N} n p_{n,0} = \frac{\mu_2 \sum_{n=1}^{N} n \prod_{i=0}^{n-1} \rho_i - p\lambda_{1,N} \prod_{i=0}^{N-1} \rho_i}{\mu_2 \sum_{n=0}^{N} \prod_{i=0}^{n-1} \rho_i}.$$

$\Longrightarrow$ N linear equations in the N unknowns $p_{1,0}, \ldots, p_{N,0}$.

Stationary quantities and numerical aspects

$$\mathbb{E}L_2 = \sum_{n=0}^{N} G_n'(1)$$

$$\mathbb{E}(L_1L_2) = \mu_2^{-1} p\lambda_{1,N}[G_N(1) + G_N'(1)]$$

Covariance of L_1 and L_2 for $N \geq 2$:

- changes signs when the system parameters are varied;
- nice function of p on the interval $[0, p^*]$: either convex or concave, either monotone decreasing or increasing or unimodal with one or two zeros (where one zero is $p = 0$);
- $\mathrm{Cov}(L_1, L_2) = 0$ for $N = 1$:

$$\mathbb{E}(L_1L_2) = \frac{\lambda_1}{\lambda_1 + \mu_1} \cdot \frac{p\lambda_1}{\mu_2 - p\lambda_1} = \mathbb{E}L_1\mathbb{E}L_2.$$

- $\mathrm{Cov}(L_1, L_2) \leq 0$ for $p = 0$ when Q_2 has its own arrival stream (Perel and Yechiali (2008)).

The model with single-customer jockeying

Limited jockeying: one customer of Q_2, if one is present, is forced to move to Q_1 as soon as Q_1 empties. This jockeying customer then starts acting as a server for Q_2.
$\longrightarrow$ Markov chain of queue lengths, which is irreducible on the state space

$$\{(0,0)\} \cup \{(n,m) \mid n = 1, \ldots, N,\ m \geq 0\}$$

The balance equations (again)

$$\lambda_{1,0}p_{0,0} = \mu_{1,1}p_{1,0}, \tag{5}$$

$$(\lambda_{1,n} + \mu_{1,n})p_{n,0} = \lambda_{1,n-1}p_{n-1,0} + \mu_{1,n+1}p_{n+1,0} + (n\mu_2 + \delta_{1n}\mu_{1,1})p_{n,1}$$
$$\text{for } n = 1, \ldots, N-1 \tag{6}$$

$$(p\lambda_{1,N} + \mu_{1,N})p_{N,0} = \lambda_{1,N-1}p_{N-1,0} + N\mu_2 p_{N,1} \tag{7}$$

$$\begin{aligned}(\lambda_{1,n} + \mu_{1,n} + n\mu_2)p_{n,m} &= \lambda_{1,n-1}p_{n-1,m}(1-\delta_{1,n}) + \mu_{1,n+1}p_{n+1,m} \\ &\quad + (n\mu_2 + \delta_{1n}\mu_{1,1})p_{n,m+1}\end{aligned} \tag{8}$$

for $n = 1, \ldots, N-1, \;\; m \geq 1$

$$\tag{9}$$

$$\begin{aligned}(p\lambda_{1,N} + \mu_{1,N} + N\mu_2)p_{N,m} &= \lambda_{1,N-1}p_{N-1,m} + p\lambda_{1,N}p_{N,m-1} \\ &\quad + N\mu_2 p_{N,m+1}, \quad m \geq 1.\end{aligned} \tag{10}$$

Stability condition

Theorem

The system of equations (5)-(10) *has a unique nonnegative and normalized solution if and only if*

$$p\lambda_{1,N}\prod_{n=1}^{N-1}\rho_n < \mu_2\sum_{n=1}^{N} n\prod_{i=1}^{n-1}\rho_i + \mu_{1,1} \qquad (11)$$

holds, where $\rho_n = \lambda_{1,n}/\mu_{1,n+1}$ *for* $n = 0, \ldots, N-1$.

Heuristic condition:

$$p\lambda_{1,N}\mathbb{P}(L_1 = N) < \mu_{1,1}\mathbb{P}(L_1 = 1) + \mu_2\mathbb{E}L_1. \qquad (12)$$

However, the distribution of L_1 is not known!
But (11) $\iff$ (12).

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However, the distribution of L_1 is not known!
But (11) $\Longleftrightarrow$ (12).

Generating functions (again)

$$G_n(z) = \sum_{m=0}^{\infty} p_{n,m} z^m, \quad |z| \leq 1.$$

Recursions:

$$\lambda_{1,0} G_0(z) = \mu_{1,1} p_{1,0} = \lambda_{1,0} p_{0,0} \tag{13}$$

$$\begin{aligned}\big(\lambda_{1,1} z - (\mu_{1,1} + \mu_2)(1-z)\big) G_1(z) \\ = \mu_{1,2} z G_2(z) - (\mu_{1,1} + \mu_2)(1-z) p_{1,0}\end{aligned} \tag{14}$$

$$\begin{aligned}\big((\lambda_{1,n} + \mu_{1,n}) z - n\mu_2(1-z)\big) G_n(z) \\ = \lambda_{1,n-1} z G_{n-1}(z) + \mu_{1,n+1} z G_{n+1}(z) - n\mu_2(1-z) p_{n,0} \\ \text{for } n = 2, \ldots, N-1\end{aligned} \tag{15}$$

$$\begin{aligned}\big(\mu_{1,N} z + (p\lambda_{1,N} z - N\mu_2)(1-z)\big) G_N(z) \\ = \lambda_{1,N-1} z G_{N-1}(z) - N\mu_2(1-z) p_{N,0}.\end{aligned} \tag{16}$$

Equations in matrix form

Define the $(N \times N)$-matrix $A(z)$ by

$$A(z) = \begin{pmatrix} \alpha_1(z) & -\mu_{1,2}z & 0 & \dots & \dots & 0 \\ -\lambda_{1,0}z & \alpha_2(z) & -\mu_{1,3}z & \ddots & & \vdots \\ 0 & \ddots & \alpha_3(z) & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & & \ddots & \ddots & \ddots & -\mu_{1,N}z \\ 0 & \dots & \dots & 0 & -\lambda_{1,N-1}z & \alpha_N(z) \end{pmatrix}, \tag{17}$$

where

$$\alpha_1(z) = \lambda_{1,1}z - (\mu_{1,1} + \mu_2)(1-z), \tag{18}$$

$$\alpha_n(z) = (\lambda_{1,n} + \mu_{1,n})z - n\mu_2(1-z) \text{ for } n = 2, \dots, N-1 \tag{19}$$

$$\alpha_N(z) = \mu_{1,N}z + (p\lambda_{1,N}z - N\mu_2)(1-z). \tag{20}$$

Equations in matrix form

$$G(z) = (G_1(z), \ldots, G_N(z))^\top,$$
$$P = ((\mu_{1,1} + \mu_2)p_{1,0}, 2\mu_2 p_{2,0}, \ldots, N\mu_2 p_{N,0})^\top$$

The equations (14), (15) and (16) are equivalent to

$$A(z)G(z) = -(1-z)P. \tag{21}$$

The roots of $\det(A(z))$

Theorem

$\det(A(z))$ *is a polynomial of degree* $N+1$ *and has* $N-1$ *distinct zeros in the interval* $(0,1)$ *and one zero at* $z=1$. *Additionally,* $\det(A(z))$ *has another zero in the interval* $(1,\infty)$ *if and only if the system of equations* (5)-(10) *has a unique nonnegative and normalized solution, i.e., if and only if* $p\lambda_{1,N}\mathbb{P}(L_1=N) < \mu_{1,1}\mathbb{P}(L_1=1)+\mu_2\mathbb{E}L_1$ *or, equivalently,*

$$p\lambda_{1,N}\prod_{n=1}^{N-1}\rho_n < \mu_2\sum_{n=1}^{N} n\prod_{i=1}^{n-1}\rho_i + \mu_{1,1}. \tag{22}$$

Exhaustive jockeying: closed-form solution

Assume, for simplicity, constant arrival and service rates in Q_1:

- Q_1: works like a $M/M/1/N-1$ queue with arrival rate $\lambda_1 > 0$ and service rate $\mu_1 > 0$.
- Q_2: fed by the p-weighted overflow stream from Q_1, $p \in [0,1]$, with service rate $n\mu_2$ as long as $L_1 = n$.
- As soon as Q_1 is not fully occupied, customers from Q_2 are instantly transferred to Q_1 until Q_1 is fully occupied or Q_2 empties, whatever happens first.

State space: $\{(N,m) \mid m \geq 0\} \cup \{(n,0) \mid 0 \leq n < N\}$.

Solution of the balance equations

$$p_{n,m} = \left(\frac{\lambda_1}{\mu_1}\right)^n \left(\frac{p\lambda_1}{\mu_1 + N\mu_2}\right)^m p_{0,0}, \quad m = 0,\ n = 0, \ldots, N-1$$

$$\text{or } m \geq 0, n = N$$

$$p_{0,0} = \left(\left(\frac{\lambda_1}{\mu_1}\right)^N \frac{\mu_1 + N\mu_2}{\mu_1 + N\mu_2 - p\lambda_1} + \frac{\mu_1 - \lambda_1 \left(\frac{\lambda_1}{\mu_1}\right)^{N-1}}{\mu_1 - \lambda_1} \right)^{-1}.$$

The necessary and sufficient stability condition is $p\lambda_1 < \mu_1 + N\mu_2$.

An alternative approach

Model the two queues as an ordinary single $M_{(n)}/M_{(n)}/1$ queue. with state space $\mathbb{Z}_+ = \{0, \ldots, N\} \cup \{N+1, N+2, \ldots\}$:

- $\{0, \ldots, N\} \quad \longleftrightarrow \quad \{(0,0), \ldots, (N,0)\}$
- $\{N+1, N+2, \ldots\} \quad \longleftrightarrow \quad \{(N,1), (N,2), (N,3), \ldots\}$
- Arrival rate: λ_1 in the states $0, \ldots, N-1$ and $p\lambda_1$ in the states $m \geq N$
- Service rate: μ_1 in the states $1, \ldots, N$ and $\mu_1 + N\mu_2$ in the states $m \geq N+1$

$\longrightarrow$ equivalent infinite birth and death chain!