Two-Queue Systems where Customers are the Servers

Uri Yechiali[†♠] (joint work with **Efrat Perel**[†])

[†] Dept. of Statistics & Operations Research, School of Mathematical Sciences, Tel Aviv University, Israel

[♠] Afeka Academic College of Engineering, Tel Aviv, Israel

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Motivation

- Signal searching in space – SETI@home project
- File sharing
- Arazi, Ben-Jacob and Yechiali [2005]: Specific example from the field of computer networks

General Structure of the Model

2 connected queueing systems

- Q_1 **is a <u>limited-buffer</u> multi-server**

  $M(\lambda_1) / M(\mu_1) / min\{L_2, N\} / N$ **system.**

  The L_2 **customers of** Q_2 **are the servers in** Q_1**.**

Q_2 **is an <u>infinite-buffer</u>** $M(\lambda_2) / M(*) / ** / \infty$ **system**

* **exponentially distributed service time.**

** **The** L_1 **customers of** Q_1 **are the servers in** Q_2 **as follows:**

2 Service Schemes for Q_2

- **Model 1: Q_2 is an $M(\lambda_2) / M(\mu_2 L_1) / 1 / \infty$ system.**

  That is, the L_1 customers of Q_1 <u>join hands together</u> and form a <u>single server</u> for the customers of Q_2.

- **Model 2: Q_2 is a <u>multi-server</u> $M(\lambda_2) / M(\mu_2) / L_1 / \infty$ system. Each one of the L_1 customers present in Q_1 acts as an individual server for the customers in Q_2.**

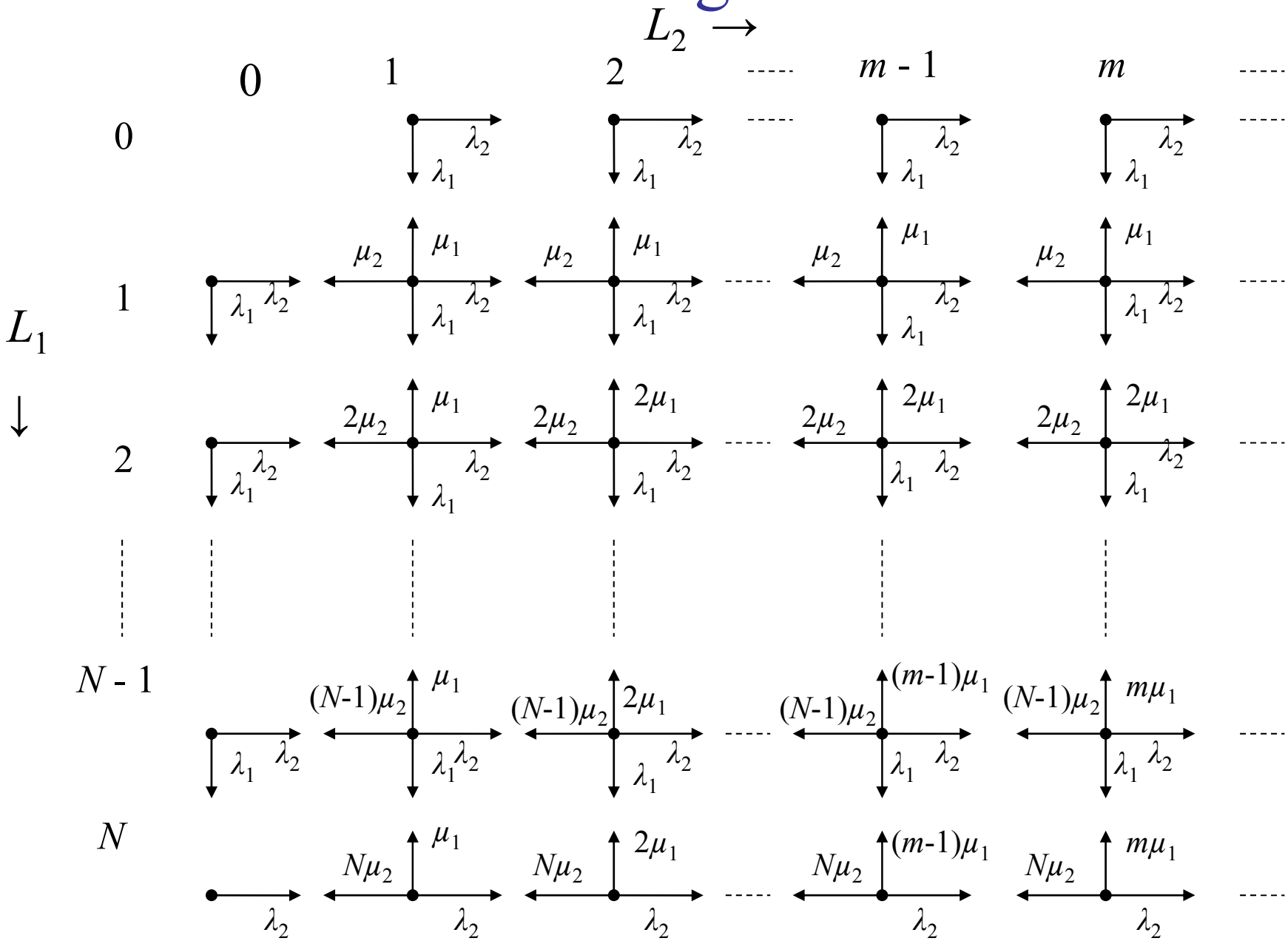
Model 1

- Q_1: $M(\lambda_1) / M(\mu_1) / min\{L_2, N\} / N$.
- Q_2: $M(\lambda_2) / M(\mu_2 L_1) / 1 / \infty$.
- L_j - Total number of customers present in Q_j, $j = 1, 2$.

- System's stationary probabilities.

$$P_{nm} = P(L_1 = n, L_2 = m), 0 \leq n \leq N, 0 \leq m.$$

Transition Rate Diagram for Model 1



Balance Equations in Matrix Geometric Form

State (n, m) - there are m jobs in Q_2 and the system is in phase n .We construct a QBD process with generator, Q

Satisfying, $\underline{P}Q = \underline{0}$

$\underline{P} = (\underline{P}_0, \underline{P}_1, \ldots, \underline{P}_m, \ldots)$

$\underline{P}_m = (P_{0m}, P_{1m}, \ldots, P_{Nm})$

$$Q = \begin{pmatrix} A_1^0 & A_0^0 & 0 & \cdots & & & & \cdots \\ A_2^1 & A_1^1 & A_0 & 0 & \cdots & & & \\ 0 & A_2 & A_1^2 & A_0 & 0 & \cdots & & \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & & \\ & \vdots & \ddots & A_2 & A_1^N & A_0 & 0 & \\ & & \vdots & \ddots & A_2 & A_1^N & A_0 & \ddots \\ & & & \vdots & \ddots & \ddots & \ddots & \ddots \\ \vdots & & & & \vdots & \ddots & \ddots & \ddots \end{pmatrix}$$

Matrix Geometric - Continued

$$A_0^0 = \begin{pmatrix} 0 & \lambda_2 & 0 & \dots & \dots & 0 \\ \vdots & 0 & \lambda_2 & 0 & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & \dots & \dots & \dots & 0 & \lambda_2 \end{pmatrix}_{N\times(N+1)}$$

$$A_0 = diag\left(\lambda_2\right)$$

Matrix Geometric - Continued

$$A_1^0 = \begin{pmatrix} -(\lambda_1+\lambda_2) & \lambda_1 & 0 & \cdots & \cdots & 0 \\ 0 & -(\lambda_1+\lambda_2) & \lambda_1 & 0 & \cdots & \vdots \\ 0 & 0 & -(\lambda_1+\lambda_2) & \lambda_1 & \ddots & \vdots \\ \vdots & 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \lambda_1 \\ 0 & \cdots & 0 & 0 & 0 & -\lambda_2 \end{pmatrix}_{N\times N}$$

Matrix Geometric - Continued

$$A_1^1 = \begin{pmatrix} -(\lambda_1+\lambda_2) & \lambda_1 & 0 & \cdots & \cdots & 0 \\ \mu_1 & -(\lambda_1+\lambda_2+\mu_1+\mu_2) & \lambda_1 & 0 & \cdots & \vdots \\ 0 & \mu_1 & -(\lambda_1+\lambda_2+\mu_1+2\mu_2) & \lambda_1 & \ddots & \vdots \\ \vdots & 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \mu_1 & -(\lambda_1+\lambda_2+\mu_1+(N-1)\mu_2) & \lambda_1 \\ 0 & \cdots & 0 & 0 & \mu_1 & -(\lambda_2+\mu_1+N\mu_2) \end{pmatrix}$$

$$A_1^2 = \begin{pmatrix} -(\lambda_1+\lambda_2) & \lambda_1 & 0 & \cdots & \cdots & 0 \\ \mu_1 & -(\lambda_1+\lambda_2+\mu_1+\mu_2) & \lambda_1 & 0 & \cdots & \vdots \\ 0 & 2\mu_1 & -(\lambda_1+\lambda_2+2\mu_1+2\mu_2) & \lambda_1 & \ddots & \vdots \\ \vdots & 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & 2\mu_1 & -(\lambda_1+\lambda_2+2\mu_1+(N-1)\mu_2) & \lambda_1 \\ 0 & \cdots & 0 & 0 & 2\mu_1 & -(\lambda_2+2\mu_1+N\mu_2) \end{pmatrix}$$

Matrix Geometric - Continued

$$A_1^3 = \begin{pmatrix} -(\lambda_1+\lambda_2) & \lambda_1 & 0 & \cdots & \cdots & 0 \\ \mu_1 & -(\lambda_1+\lambda_2+\mu_1+\mu_2) & \lambda_1 & 0 & \cdots & \vdots \\ 0 & 2\mu_1 & -(\lambda_1+\lambda_2+2\mu_1+2\mu_2) & \lambda_1 & \ddots & \vdots \\ \vdots & 0 & 3\mu_1 & -(\lambda_1+\lambda_2+3\mu_1+3\mu_2) & \lambda_1 & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \lambda_1 \\ 0 & \cdots & 0 & 0 & 3\mu_1 & -(\lambda_2+3\mu_1+N\mu_2) \end{pmatrix}$$

$$\vdots$$

$$A_1^N = \begin{pmatrix} -(\lambda_1+\lambda_2) & \lambda_1 & 0 & \cdots & \cdots & 0 \\ \mu_1 & -(\lambda_1+\lambda_2+\mu_1+\mu_2) & \lambda_1 & 0 & \cdots & \vdots \\ 0 & 2\mu_1 & -(\lambda_1+\lambda_2+2\mu_1+2\mu_2) & \lambda_1 & \ddots & \vdots \\ \vdots & 0 & 3\mu_1 & -(\lambda_1+\lambda_2+3\mu_1+3\mu_2) & \lambda_1 & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \lambda_1 \\ 0 & \cdots & 0 & 0 & N\mu_1 & -(\lambda_2+N\mu_1+N\mu_2) \end{pmatrix}$$

Matrix Geometric - Continued

$$A_2^1 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ \mu_2 & 0 & \dots & \dots & \vdots \\ 0 & 2\mu_2 & 0 & \dots & \vdots \\ \vdots & \ddots & 3\mu_2 & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ 0 & \dots & \dots & 0 & N\mu_2 \end{pmatrix}_{(N+1)\times N} \qquad A_2 = \begin{pmatrix} 0 & \dots & \dots & \dots & \dots & 0 \\ \vdots & \mu_2 & 0 & \dots & \dots & 0 \\ \vdots & 0 & 2\mu_2 & \ddots & \dots & \vdots \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & \dots & \dots & \dots & \dots & N\mu_2 \end{pmatrix}$$

Matrix Geometric - Continued

- *Let* $A_1 = A_1^N$
- $A = A_0 + A_1 + A_2$ - Infinitesimal generator of the $M(\lambda_1) / M(\mu_1) / N / 0$ model
- $\underline{\pi} = (\pi_0, \pi_1, \ldots, \pi_N)$ - stationary vector of the matrix A, $\underline{\pi} A = \underline{0}, \quad \underline{\pi} \cdot \underline{e} = 1$

Matrix Geometric - Continued

- System's stability condition

$$\underline{\pi}\, A_0\, \underline{e} < \underline{\pi}\, A_2\, \underline{e}$$

- Resulting in

$$\lambda_2 < \mu_2 \frac{\sum_{n=0}^{N} \left(\frac{\lambda_1}{\mu_1}\right)^n \frac{1}{(n-1)!}}{\sum_{n=0}^{N} \left(\frac{\lambda_1}{\mu_1}\right)^n \frac{1}{n!}} = \mu_2 E[L_{M(\lambda_1)/M(\mu_1)/N/0}]$$

Matrix Geometric - Continued

- $\underline{P}_m = (P_{0m}, P_{1m}, \ldots, P_{Nm})$ - Steady state probability vector.
- $\underline{P}_m = \underline{P}_{N-1} R^{m-(N-1)}, m \geq N.$
- R is the minimal nonnegative solution of the matrix quadratic equation

$$A_0 + RA_1 + R^2 A_2 = 0$$

Matrix Geometric - Continued

- $\underline{P}_0, \underline{P}_1, \ldots, \underline{P}_{N-1}$ are derived by solving the linear system,

$$\underline{P}_0 A_1^0 + \underline{P}_1 A_2^1 = \underline{0}$$

$$\underline{P}_0 A_0^0 + \underline{P}_1 A_1^1 + \underline{P}_2 A_2 = \underline{0}$$

$$\underline{P}_{m-1} A_0 + \underline{P}_m A_1^m + \underline{P}_{m+1} A_2 = \underline{0} \quad , 2 \le m \le N-2$$

$$\underline{P}_{N-2} A_0 + \underline{P}_{N-1} \left(A_1^{N-1} + R A_2 \right) = \underline{0}$$

$$\sum_{m=0}^{N-2} \underline{P}_m \underline{e} + \underline{P}_{N-1} \left[I - R \right]^{-1} \underline{e} = 1$$

Matrix Geometric - Continued

- Mean total number of customers in Q_2,

$$E[L_2] = \sum_{m=0}^{\infty} m\underline{P}_m \cdot \underline{e} = \sum_{m=0}^{N-2} m\underline{P}_m \cdot \underline{e} + \sum_{m=N-1}^{\infty} m\underline{P}_{N-1} R^{m-N+1} \cdot \underline{e}$$

$$= \sum_{m=0}^{N-2} m\underline{P}_m \cdot \underline{e} + (N-2)\underline{P}_{N-1}\left[I-R\right]^{-1} \cdot \underline{e} + \underline{P}_{N-1}\left[I-R\right]^{-2} \cdot \underline{e}$$

An Alternative Approach
Generating Functions

- A system of linear equations,

$$A(z)\underline{G}(z) = \underline{P}(z)$$

where,

$$\underline{G}(z) = \left(G_0(z), G_1(z), \ldots, G_N(z)\right)^t$$

$$G_n(z) = \sum_{m=0}^{\infty} P_{nm} z^m, \; 0 \le n \le N$$

Generating Functions - Continued

$$P_n(z) = \begin{cases} -\mu_1 P_{10} & ,n = 0 \\ \mu_1 z\left(\sum_{m=0}^{n-1}(n-m)z^m P_{nm} - \sum_{m=0}^{n}(n+1-m)z^m P_{n+1,m}\right) & \\ \qquad -n\mu_2(1-z) & ,1 \le n \le N-1 \\ \mu_1 z\sum_{m=0}^{N-1}(N-m)z^m P_{Nm} - N\mu_2(1-z) & ,n = N \end{cases}$$

Generating Functions - Continued

$$A_{(N+1)\times(N+1)}(z) = \begin{pmatrix} \alpha_0^{(N)}(z) & -\mu_1 & 0 & \cdots & \cdots & 0 \\ -\lambda_1 z & \alpha_1^{(N)}(z) & -2\mu_1 z & 0 & \cdots & 0 \\ 0 & \ddots & \alpha_2^{(N)}(z) & \ddots & 0 & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & -N\mu_1 z \\ 0 & \cdots & \cdots & 0 & -\lambda_1 z & \alpha_N^{(N)}(z) \end{pmatrix}$$

$$\alpha_0^{(N)}(z) = \lambda_1 + \lambda_2(1-z)$$

$\underline{1 \le n \le N-1:}$ $$\alpha_n^{(N)}(\mathrm{z}) = (\lambda_1 + n\mu_1)\mathrm{z} + (\lambda_2 \mathrm{z} - n\mu_2)(1-z)$$

$$\alpha_N^{(N)}(z) = N\mu_1 z + (\lambda_2 z - N\mu_2)(1-z)$$

Generating Functions - Continued

To derive the PGFs we need to find the $N(N+1)/2$ boundary probabilities, P_{10}; P_{20}, P_{21}; …; $P_{N\,0}$, $P_{N\,1}$, $P_{N,\,N\text{-}1}$, appearing in $\underline{P}(z)$.

By Cramer's Rule:

$$G_n(z) = \frac{|A_n(z)|}{|A(z)|}$$

Derivation of the Boundary Probabilities

Theorem: For any $\lambda_1 > 0$, μ_1, $\lambda_2 \geq 0$, $\mu_2 > 0$ and $N \geq 1$, $|A(z)|$ is a polynomial of degree $2N + 1$ possessing N -1 <u>distinct</u> roots in the open interval (0,1), a single root at $z = 1$, and N roots in the open interval $(1, \infty)$. Another root exists in the open interval (0,1) provided that

$$\lambda_2 > \mu_2 E\,[L(M(\lambda_1) / M(\mu_1) / N / 0)]$$

We derive the probabilities by utilizing the N - 1 roots in (0,1) and by using some of the balance equations.

Covariance

- $Cov\left(L_1, L_2\right) = \dfrac{\lambda_2}{\mu_2} - E[L_2]\left(E[L_1] - \dfrac{\lambda_2}{\mu_2}\right)$

- We argue that in steady state $Cov\ (L_1, L_2) \leq 0$.

  We thus get a lower bound for $E[L_2]$:

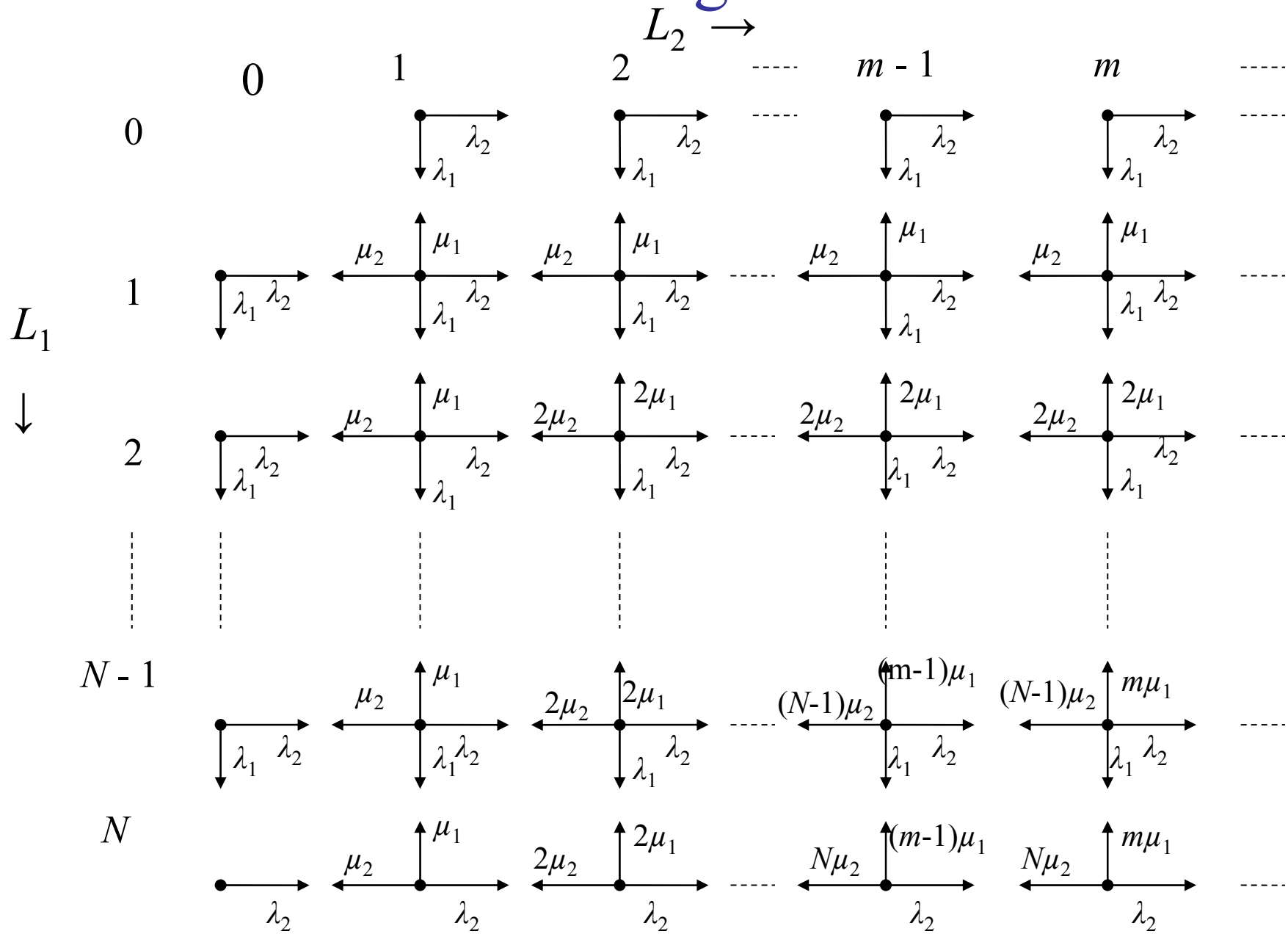
$$E[L_2] \geq \frac{\dfrac{\lambda_2}{\mu_2}}{E[L_1] - \dfrac{\lambda_2}{\mu_2}} = \frac{\lambda_2}{\mu_2 E[L_1] - \lambda_2} = E[L(M(\lambda_2)/M(\mu_2 E[L_1])/1)]$$

Model 2

- Q_1 operates as in Model 1:
  $M(\lambda_1) / M(\mu_1) / min\{L_2, N\} / N$

- Q_2 : Each customer present in Q_1 acts as an <u>individual</u> <u>server</u>. That is, Q_2 is an $M(\lambda_2) / M(\mu_2) / L_1 / \infty$ system with dynamically changing L_1.

Transition Rate Diagram for Model 2



Stability Condition

Using similar analytical approach we arrive at the <u>same stability condition</u>, and derive the system's stationary probabilities.

Models 3 & 4

- Model 3

  Q_1 operates as a limited buffer <u>single-server</u> $M(\lambda_1) / M(\mu_1 L_2) / 1 / N\text{-}1$ queue, while Q_2 is an $M(\lambda_2) / M(\mu_2 L_1) / 1 / \infty$ system.

- Model 4

  Q_1 operates as in Model 3, but Q_2 is an $M(\lambda_2) / M(\mu_2) / L_1 / \infty$ system.

Models 3 & 4

- For these models using PGF's leads to a set of unsolvable differential equations.

- Matrix Geometric can not be used since there is no geometric structure.