On the damped geometric telegrapher's process

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Abstract The geometric telegrapher's process has been proposed in 2002 as a model to describe the dynamics of the price of risky assets. In this contribution we consider a similar stochastic process, for which the random times between consecutive slope changes have exponential distribution with linearly increasing parameters. This leads to a process characterized by a damped behavior. We study the main features of the transient probability law of the process, and of its stationary limit.

2 The stochastic model and probability laws

Let us assume that the price of risky assets is described by the following stochastic process, that we shall call damped geometric telegrapher's process:

$$S_t = s_0 \exp\left[a t + X_t\right], \qquad \text{with} \quad X_t = c \int_0^t (-1)^{N_\tau} \mathrm{d}\tau, \quad t \geq 0, \tag{2}$$

where $s_0 > 0$, $a \in \mathbb{R}$, $c > 0$, and where N_t is an alternating counting process characterized by independent random times U_k, D_k, $k \geq 1$. Hence,

$$N_0 = 0, \qquad N_t = \sum_{n=1}^{\infty} \mathbf{1}_{\{T_n \leq t\}}, \qquad t > 0,$$

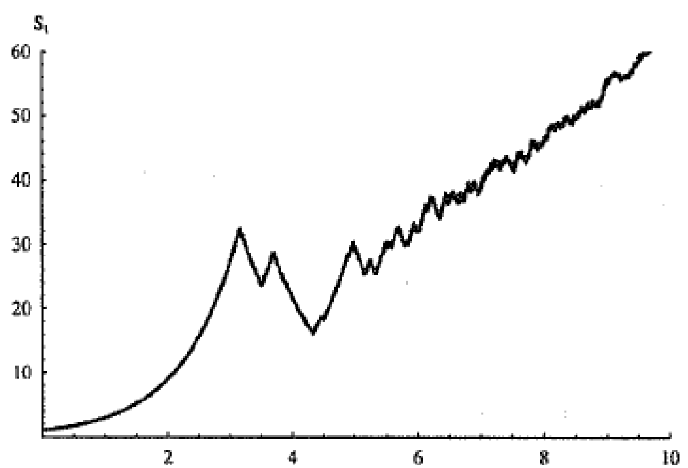
where $T_{2k} = U^{(k)} + D^{(k)}$ and $T_{2k+1} = T_{2k} + U_{k+1}$ for $k = 0, 1, \ldots$, with $U^{(0)} = D^{(0)} = 0$ and

$$U^{(k)} = U_1 + U_2 + \cdots + U_k, \qquad D^{(k)} = D_1 + D_2 + \cdots + D_k, \qquad k = 1, 2, \ldots. \tag{3}$$

We assume that $\{U_k\}$ and $\{D_k\}$ are mutually independent sequences of independent random variables characterized by exponential distribution with parameters

$$\lambda_k = \lambda\, k, \qquad \mu_k = \mu\, k, \qquad (\lambda, \mu > 0,\ k = 1, 2, \ldots), \tag{4}$$

respectively. We remark that process S_t has bounded variations and its sample-paths are constituted by connected lines having exponential behavior, characterized alternately by growth rates $a + c$ and $a - c$, where a is the growth rate of risky assets' price in the absence of randomness, and c is the intensity of the random factor of alternating type. Assumption (4) implies that the reversal rates λ_k and μ_k linearly increase with the number of reversals, so that the sample paths of S_t are subject to an increasing number of slope changes when t increases, this giving a damped behavior. (An example is shown in Fig. 1).



Proposition 1. *For $k = 1, 2, \ldots$ we have*

$$F^{(k)}(u) := P(U^{(k)} \le u) = (1 - e^{-\lambda u})^k, \qquad u \ge 0. \tag{5}$$

Proof. The proof proceeds by induction on k. For $k = 1$, the result is obvious. Let us now assume (5) holding for all $m = 1, \ldots, k-1$. Denoting by $\widetilde{U}_j$ $(j \ge 1)$ independent and exponentially distributed random variables with parameters λ, due to (4) and recalling (3) we have

$$U^{(k)} \stackrel{\mathrm{D}}{=} \sum_{j=1}^{k} \frac{\widetilde{U}_j}{j}, \qquad k = 1, 2, \ldots.$$

Then, due to independence, we have

$$\begin{aligned} F^{(k)}(u) &= \lambda k \int_0^u e^{-\lambda k y} (1 - e^{-\lambda (u-y)})^{k-1} dy \\ &= \lambda k \sum_{j=0}^{k-1} (-1)^j \binom{k-1}{j} e^{-\lambda j u} \int_0^u e^{-\lambda (k-j) y} dy \\ &= \frac{k}{k-j} \sum_{j=0}^{k-1} (-1)^j \binom{k-1}{j} e^{-\lambda j u} [1 - e^{-\lambda (k-j) u}] \\ &= \sum_{j=0}^{k-1} (-1)^j \binom{k}{j} [e^{-\lambda j u} - e^{-\lambda k u}] = \sum_{j=0}^{k-1} (-1)^j \binom{k}{j} e^{-\lambda j u} + (-1)^k e^{-\lambda k u} \\ &= (1 - e^{-\lambda u})^k, \end{aligned}$$

this giving (5).

and μ, respectively.

In order to obtain the distribution function of process X_t, let us now introduce the compound process

$$Y_t = \sum_{n=0}^{M_t} D_n, \qquad \text{where} \quad M_t := \max\{n \geq 0 : \sum_{j=1}^{n} U_j \leq t\}, \qquad t > 0.$$

Notice that $P(Y_t = 0) = \mathrm{e}^{-\lambda t}$. Hereafter we obtain the distribution function of Y_t.

Proposition 2. *For any fixed $t > 0$ and $y \in (0, +\infty)$, we have*

$$H(y,t) := P(Y_t \leq y) = \frac{\mathrm{e}^{-\lambda t}}{\mathrm{e}^{-\lambda t} + \mathrm{e}^{-\mu y}(1 - \mathrm{e}^{-\lambda t})}. \tag{7}$$

Proof. For $t > 0$ the distribution function of Y_t can be expressed as

$$H(y,t) = \sum_{n=0}^{+\infty} P(M_t = n)\, G^{(n)}(y),$$

where, due to (5),

$$P(M_t = n) = F^{(n)}(t) - F^{(n+1)}(t) = \mathrm{e}^{-\lambda t}(1 - \mathrm{e}^{-\lambda t})^n, \quad n = 0, 1, \ldots.$$

Hence, recalling (6), we obtain

$$H(y,t) = \mathrm{e}^{-\lambda t} \sum_{n=0}^{+\infty} (1 - \mathrm{e}^{-\lambda t})^n (1 - \mathrm{e}^{-\mu y})^n,$$

$$W_t = \int_0^t \mathbf{1}_{\{N_s \text{ even}\}} \mathrm{d}s, \qquad t > 0,$$

so that

$$X_t = c(2W_t - t), \qquad t > 0. \tag{8}$$

Proposition 3. *For all $0 < \tau < t$, the distribution function of W_t is:*

$$P(W_t \le \tau) = \frac{\mathrm{e}^{-\mu(t-\tau)}(1-\mathrm{e}^{-\lambda\tau})}{\mathrm{e}^{-\lambda\tau} + \mathrm{e}^{-\mu(t-\tau)}(1-\mathrm{e}^{-\lambda\tau})}. \tag{9}$$

Moreover,

$$P(W_t < t) = 1 - \mathrm{e}^{-\lambda t}, \qquad P(W_t \le t) = 1.$$

Proof. Note that, for a fixed value $t_0 > 0$,

$$W_{t_0} = \inf\{t > 0 : Y(t) \ge t_0 - t\}. \tag{10}$$

Moreover, if $W_{t_0} = \tau$, $\tau \le t_0$, and $Y_\tau = t_0 - \tau$ ($Y_\tau > t_0 - \tau$), then the motion is moving upward (downward) at time t_0. Finally, since Y_t is an increasing process, due to (10), the survival distribution of W_t is given by

$$P(W_t > \tau) = H(t-\tau, \tau), \qquad 0 < \tau \le t,$$

so that Eq. (9) follows.

Proposition 4. *Let $\tau_* = \tau_*(x,t) = (x+ct)/(2c)$. For all $t > 0$ and $x < ct$ we have*

$$P(X_t \le x) = \frac{\mathrm{e}^{-\mu(t-\tau_*)}(1-\mathrm{e}^{-\lambda\tau_*})}{\mathrm{e}^{-\lambda\tau_*} + \mathrm{e}^{-\mu(t-\tau_*)}(1-\mathrm{e}^{-\lambda\tau_*})}.$$

Moreover, $P(X_t < ct) = 1-\mathrm{e}^{-\lambda t}$ and $P(X_t \le ct) = 1$.

In the following proposition we finally obtain the distribution function of S_t.

Proposition 5. *For all $t > 0$ and $x < s_0\,\mathrm{e}^{(a+c)t}$, we have*

$$P(S_t \le x) = \frac{A_\mu(t)\,[x/s_0]^{(\lambda+\mu)/(2c)} - A_\mu(t)A_\lambda(t)\,[x/s_0]^{\mu/(2c)}}{A_\lambda(t) + A_\mu(t)\,[x/s_0]^{(\lambda+\mu)/(2c)} - A_\mu(t)A_\lambda(t)\,[x/s_0]^{\mu/(2c)}},$$

where $A_\lambda(t) = \exp\left\{-\frac{\lambda}{2}\left(1-\frac{a}{c}\right)t\right\}$ and $A_\mu(t) = \exp\left\{-\frac{\mu}{2}\left(1+\frac{a}{c}\right)t\right\}$. Moreover,

$$P(S_t < s_0\,\mathrm{e}^{(a+c)t}) = 1-\mathrm{e}^{-\lambda t}, \qquad P(S_t \le s_0\,\mathrm{e}^{(a+c)t}) = 1.$$

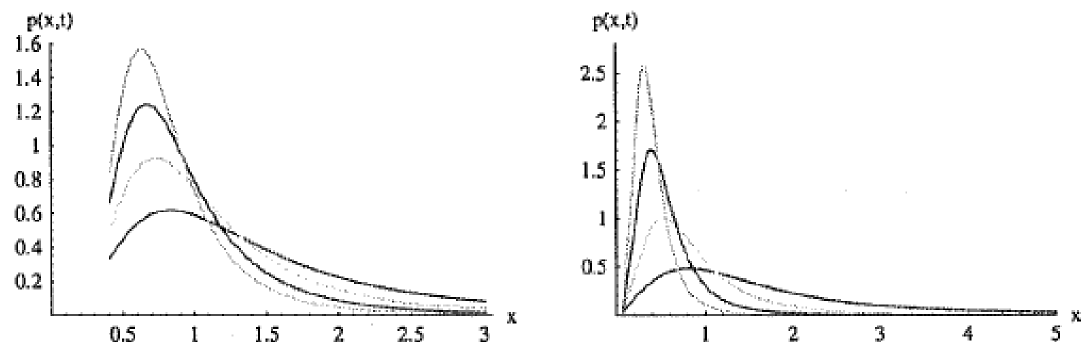


Fig. 2 Plot of $p(x,t)$ for $s_0 = 1$, $a = 0.1$, $c = 1$, $\mu = 2$, and $\lambda = 2,3,4,5$, from bottom to top near the origin, with $t = 1$ (left-hand-side) and $t = 3$ (right-hand-side).

Proposition 6. *For all $t > 0$ we have*

$$P\{S_t = s_0\,\mathrm{e}^{(a+c)t}\} = \mathrm{e}^{-\lambda t},$$

and, for $x \in (s_0\,\mathrm{e}^{(a-c)t}, s_0\,\mathrm{e}^{(a+c)t})$

$$p(x,t) := \frac{\mathrm{d}}{\mathrm{d}x}P(S_t \le x) = \frac{\lambda + \mu - \mu\left(\frac{x}{s_0}\right)^{-\frac{\lambda}{2c}}\exp\{-\frac{\lambda t(c-a)}{2c}\}}{2cx}$$
$$\times \frac{1}{\{2\cosh\{\frac{\lambda+\mu}{4c}\log\left(\frac{x}{s_0}\right) + \frac{\lambda t(c-a)-\mu t(c+a)}{4c}\} - \left(\frac{x}{s_0}\right)^{-\frac{\lambda-\mu}{4c}}\exp\{-\frac{\lambda t(c-a)+\mu t(c+a)}{4c}\}\}^2}.$$

Some plots of density $p(x,t)$ are shown in Fig. 2 for various choices of λ and t. Let us now analyze the behavior of $p(x,t)$ in the limit as t tends to $+\infty$.

Corollary 1. *If $\lambda(c-a) = \mu(c+a)$ then*

$$\lim_{t\to+\infty} p(x,t) = \frac{\beta}{s_0}\frac{(x/s_0)^{\beta-1}}{[1+(x/s_0)^{\beta}]^2}, \qquad x \in (0,+\infty),$$

where $\beta = \lambda/(c+a)$; whereas, if $\lambda(c-a) \neq \mu(c+a)$ then

$$\lim_{t\to+\infty} p(x,t) = 0.$$

Corollary 2. *Let* $\alpha_t = s_0 \exp\{at\}$. *If* $\lambda = \mu \to +\infty$, $c \to +\infty$, *with* $\lambda/c \to \theta$, *then*

$$p(x,t) \to \frac{\theta}{\alpha_t} \frac{(x/\alpha_t)^{\theta-1}}{[1+(x/\alpha_t)^{\theta}]^2}, \qquad x \in (0,+\infty).$$

We now analyse the behavior of $p(x,t)$ when x approaches the endpoints of its support, i.e. the interval $[s_1, s_2] := [s_0\,\mathrm{e}^{(a-c)t}, s_0\,\mathrm{e}^{(a+c)t}]$.

Corollary 3. *For any fixed* $t > 0$, *we have*

$$\lim_{x \downarrow s_1} p(x,t) = \frac{\lambda}{2cs_0}\mathrm{e}^{(c-a-\mu)t}, \qquad \lim_{x \uparrow s_2} p(x,t) = \frac{[\lambda + \mu(1-\mathrm{e}^{-\lambda t})]\,\mathrm{e}^{-(c+a+\lambda)t}}{2cs_0}.$$

Hereafter we express the m-th moment of S_t in terms of the Gauss hypergeometric function ${}_2F_1$.

Proposition 7. *Let m be a positive integer. Then, for $t > 0$,*

$$E[S_t^m] = s_0^m \,\mathrm{e}^{m(a-c)t}\left\{1+\frac{2mc}{\lambda}\sum_{k=0}^{+\infty}\frac{(1-\mathrm{e}^{-\lambda t})^{k+1}}{k+1}\right.$$
$$\left.\times\sum_{r=0}^{k}\binom{k}{r}(-\mathrm{e}^{-\mu t})^r {}_2F_1\left(\frac{2mc}{\lambda}+\frac{\mu}{\lambda}r,k+1;k+2;1-\mathrm{e}^{-\lambda t}\right)\right\}. \quad (11)$$

Proof. Due to Proposition 4, by setting $y=(ct+x)/2c$ we have

$$M_{X_t}(s) := E\left[\mathrm{e}^{sX(t)}\right] = \mathrm{e}^{-sct}\left\{1+2sc\int_0^t \frac{\mathrm{e}^{-(\lambda-2cs)y}}{\mathrm{e}^{-\lambda y}+\mathrm{e}^{-\mu(t-y)}(1-\mathrm{e}^{-\lambda y})}\,\mathrm{d}y\right\}. \quad (12)$$

After some calculations Eq. (12) becomes

$$M_{X_t}(s) = \mathrm{e}^{-sct}\left\{1+\frac{2sc}{\lambda}\sum_{k=0}^{+\infty}\sum_{r=0}^{k}\binom{k}{r}(-\mathrm{e}^{-\mu t})^{k-r}\int_{\mathscr{I}} x^k(1-x)^{-[2cs+\mu(k-r)]/\lambda}\,\mathrm{d}x\right\},$$

where $\mathscr{I}=(0,1-\mathrm{e}^{-\lambda t})$. Hence, recalling Eq. (3.194.1) of [5], and noting that $E[S_t^m]=s_0^m\,\mathrm{e}^{mat}M_{X_t}(m)$ due to (2), the right-hand-side of (11) immediately follows.

Remark 1. If $\lambda=\mu$, then the moment (11) can be expressed as:

$$E[S_t^m] = s_0^m \,\mathrm{e}^{m(a-c)t}\left\{1+\frac{2mc}{\lambda}\sum_{k=0}^{+\infty}\frac{(k!)^2(1-\mathrm{e}^{-\lambda t})^{2k+1}}{(2k+1)!}\right.$$
$$\left.\times\ {}_2F_1\left(\frac{2mc}{\lambda}+k,k+1;2k+2;1-\mathrm{e}^{-\lambda t}\right)\right\}.$$

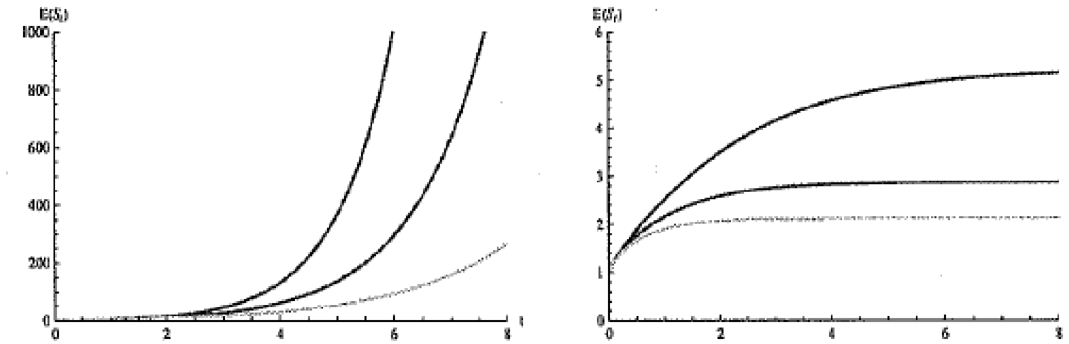


Fig. 3 Plot of $E(S_t)$ for $(\lambda,\mu)=(1.5,0.9),(1.75,1.05),(2,1.2)$ (left-hand-side) and for $(\lambda,\mu)=(3,1.8),(3.5,2.1),(4,2.4)$ (right-hand-side) from top to bottom, with $s_0=1$, $a=0.5$, $c=2$.

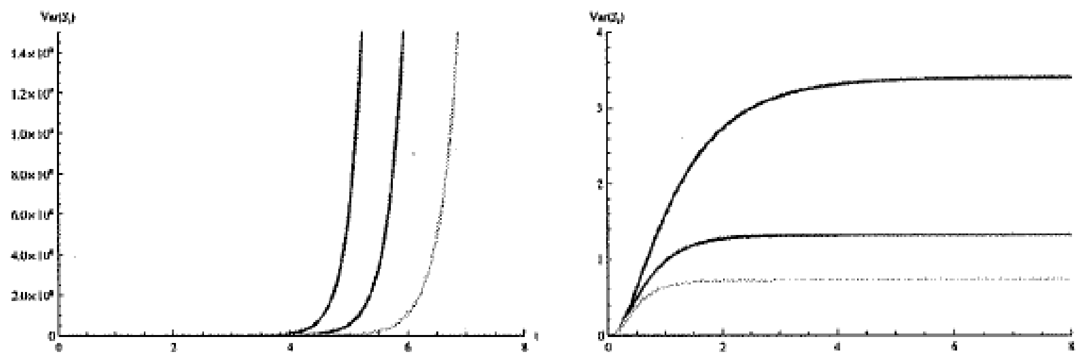


Fig. 4 Plot of $\mathrm{Var}(S_t)$ for $(\lambda,\mu)=(1,0.6),(1.5,0.9),(2,1.2)$ (left-hand-side) and for $(\lambda,\mu)=(6,3.6),(7,4.2),(8,4.8)$ (right-hand-side) from top to bottom, with $s_0=1$, $a=0.5$, $c=2$.

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