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Mathematics converts multi-item multi-echelon networks into serial multi-echelon systems

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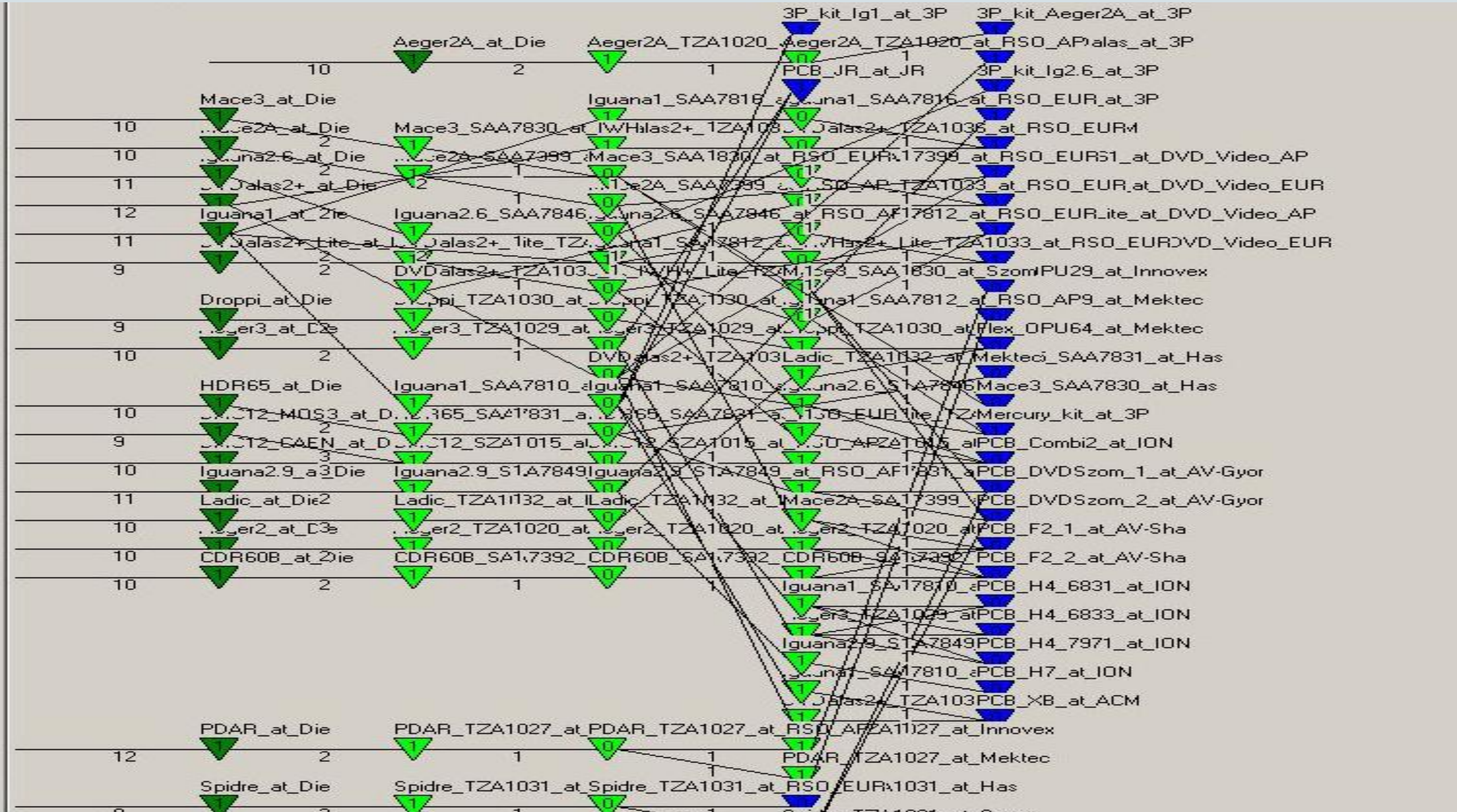
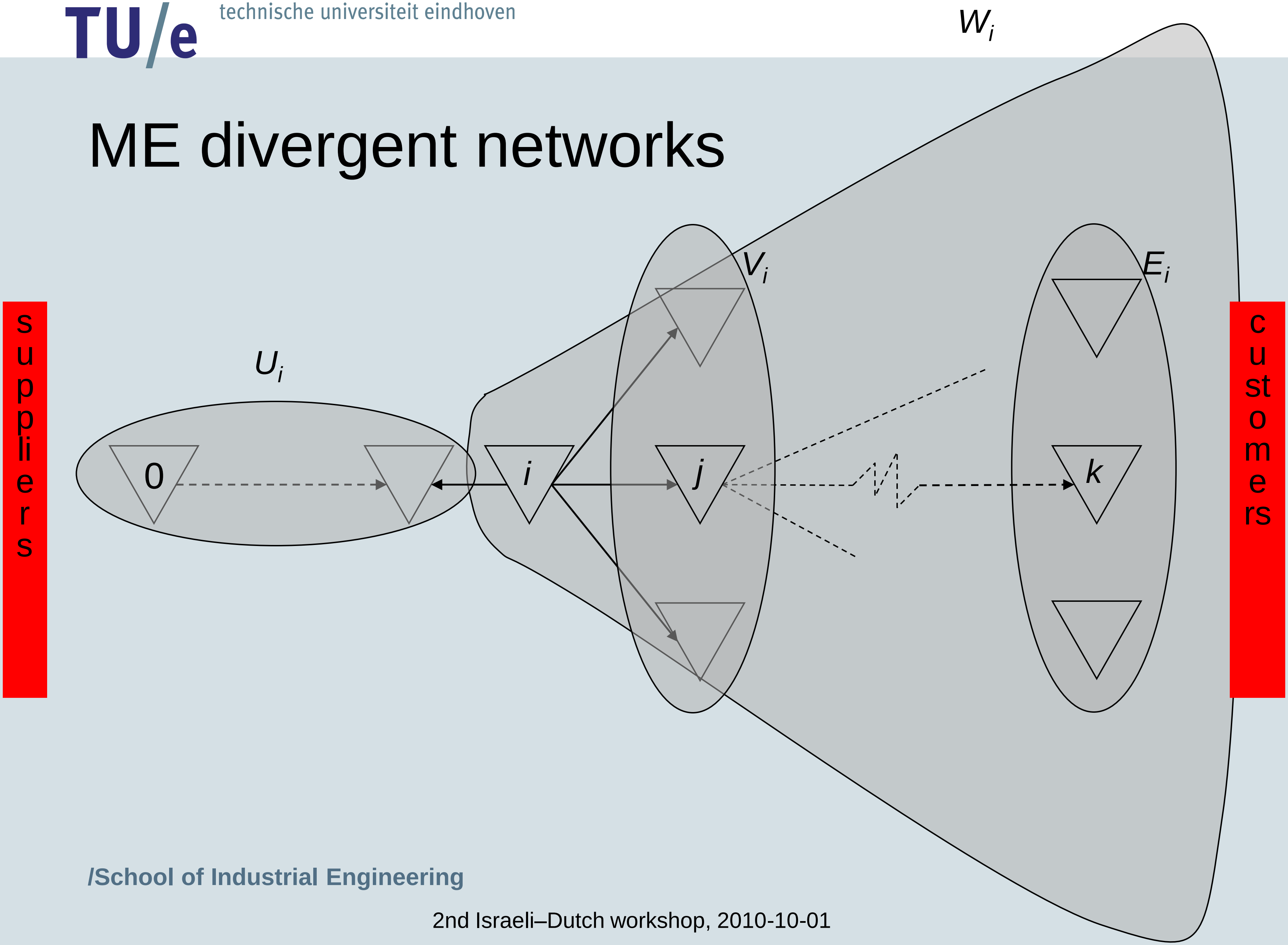
Real-life supply "chain"



ME divergent networks



Definitions

h_i	added value created (holding cost) at item i
p_k	penalty cost per unit short of end-item k
U_i	set of predecessors of i, i.e. all child items
V_i	set of successors of i, i.e. all child items
W_i	set of all items in the echelon of i
E_i	set of all end-items in the echelon of i
Y_i	echelon inventory position of i
X_i	net stock of item i
I_{ki}	net stock of end-item k in a system with root node i

Base stock policies and linear allocation rules

Shortfall of i

Parameter that determines stock kept at i on behalf of j

$$Y_j(t) = S_j - q_j\left(Z_i(t-L_i) + D_i\left(t-L_i, t\right] - \Delta_j\right)^+$$

End-item demand during lead time of i

$$S_i = \sum_{j \in V_i}\left(S_j + q_j \Delta_j\right)$$

Key recursion

$$Z_j(t) = q_j\left(Z_i(t-L_i) + D_i\left(t-L_i, t\right] - \Delta_j\right)^+$$

Allocation fractions

- Eppen and Schrage (1981): equal stockout probability

$$q_j = \frac{\sigma_j}{\sum_{m \in V_i} \sigma_m}$$

- Van der Heijden et al (1997): minimal imbalance

$$q_j = \frac{\sigma_j^2}{2\sum_{j \in V_i} \sigma_m^2} + \frac{\mu_j^2}{2\sum_{j \in V_i} \mu_m^2}$$

Optimization problem

- Given allocation fractions, determine base stock levels for all items
- Expressions for costs and performance are similar to expressions for serial systems
- Imbalance should be sufficiently low to ensure validity of analytical results
  - in case imbalance too high, base stock levels should be such that average stocks are increased at upstream stages with high imbalance
  - requires experimental research

Optimal order-up-to-policies for divergent MEIS:

Generalized Newsvendor equations theorem for periodic review systems without setup costs

Optimal order-up-to-policies and allocation functions under balance assumption satisfy generalized Newsvendor equations:

Probability of non-stockout at downstream stockpoint k in the divergent subsystem with stockpoint i as most upstream stage equals

$$P\{I_{ki} \geq 0\} = \frac{p_k + \sum_{j \in U_i} h_j}{p_k + h_k + \sum_{j \in U_k} h_j}$$

or equivalently

$$\left(p_k + \sum_{j \in U_i} h_j\right) = \left(p_k + h_k + \sum_{j \in U_k} h_j\right) P\{I_{ki} \geq 0\}$$

Sample path condition on control policies

Assume that each node i is controlled according to a policy, which is defined by a parameter ξ_i and possibly other parameters and functions

Sample path condition

If ξ_j is increased by $|E_j|\varepsilon$ for all $j \in W_j$, and there is upstream availability, then a one-time additional flow of ε units is created towards each $k \in E_j$

Main theorem

- If sample path condition holds then $\forall i \in I$

$$\sum_{k \in E_i} \left(p_k + \sum_{j \in U_k \setminus W_i} h_j \right) = \sum_{k \in E_i} \left(p_k + h_k + \sum_{j \in U_k} h_j \right) P\left\{ I_{ki} \geq 0 \right\}$$

- Proof based on echelon costs and similar to proof of single-item Newsvendor equation
- Relationship can be used as heuristic

Net stocks of end-items

Key recursion

$$Z_j(t) = q_j\left(Z_i(t-L_i) + D_i\left(t-L_i, t\right] - \Delta_j\right)^+$$

with

$$D_{L_i} = D_i\left(t-L_i, t\right]$$

results into

$$I_{ki} = S_k - \left\{\left(\ldots q_n\left(q_j\left(D_{L_i} - \Delta_i\right)^+ + D_{L_j} - \Delta_j\right)^+ + \ldots\right)^+ + D_{L_k+1}\right\}, k \in E_n, n \in V_j, j \in V_i$$

and

$$P\left\{I_{ki} \geq 0\right\} = P\left\{\left(\left(\ldots q_n\left(q_j\left(D_{L_i} - \Delta_i\right)^+ + D_{L_j} - \Delta_j\right)^+ + \ldots\right)^+ + D_{L_k+1} - S_k\right)^+ = 0\right\}, k \in E_n, n \in V_j, j \in V_i$$

Ruin probabilities and generalized Newsboy equations

Let $X_1, \dots, X_N$ be independent random variables and $x_1, \dots, x_N$ be non-negative real numbers. Then

$$P\left\{\left(\left(\left((X_N - x_N)^+ + X_{N-1} - x_{N-1}\right)^+ + \dots\right)^+ + X_1 - x_1\right)^+ = 0\right\} = P\left\{X_1 \le x_1, X_1 + X_2 \le x_1 + x_2, \dots, \sum_{i=1}^{N} X_i \le \sum_{i=1}^{N} x_i\right\}$$

I.e. non-stockout probabilities in multi-echelon order-up-to-systems are equivalent to general finite horizon ruin probabilities

Finite horizon ruin probabilities

- Non-stockout probabilities $P\{I_{ki} \geq 0\}$ can be written as finite horizon ruin probabilities

$$P\left\{I_{ki} \geq 0\right\} = P\left\{\sum_{k=1}^{j} X_k \leq \xi_j,\ j = 1, \ldots, i\right\}$$

- Expressions can be accurately approximated recursively

$$G_i(\xi_1, \ldots, \xi_i) = P\left\{\sum_{k=1}^{j} X_k \leq \xi_j,\ j = 1, \ldots, i\right\}$$

Recursive computations

- Define random variables Y_i

$$P\{Y_1 \le x\} = P\{X_1 \le x\}$$

$$P\{Y_i \le x\} = \frac{G_i(\xi_1,\ldots,\xi_{i-1},x)}{G_{i-1}(\xi_1,\ldots,\xi_{i-1})}, x \ge 0,\ i=2,\ldots,N.$$

- Theorem

$$P\{Y_i \le x\} = \frac{P\{X_i + Y_{i-1} \le x, Y_{i-1} \le \xi_{i-1}\}}{P\{Y_{i-1} \le \xi_{i-1}\}}, i = 2,\ldots,N$$

Two-moment recursion

- Fit mixture of Erlang distributions on $E[Y_i]$ and $\sigma^2(Y_i)$

$$E\left[Y_i\right] = E\left[X_i\right] + E\left[Y_{i-1} \middle| Y_{i-1} \le \xi_{i-1}\right],\ i = 2,\dots,N\text{ -1}$$

$$\sigma^2\left(Y_i\right) = \sigma^2\left(X_i\right) + \sigma^2\left(Y_{i-1} \middle| Y_{i-1} \le \xi_{i-1}\right),\ i = 2,\dots,N\text{ -1}$$

- Compute $P\{I_{ki} \geq 0\} = G_i(\xi_1,\dots,\xi_i)$ recursively

$$G_1\left(\xi_1\right) = P\left\{Y_1 \le \xi_1\right\}$$

$$G_i\left(\xi_1,\dots,\xi_i\right) = P\left\{Y_i \le \xi_i\right\} G_{i-1}\left(\xi_1,\dots,\xi_{i-1}\right), i = 2,\dots,N$$

Main theorem

- If sample path condition holds then $\forall i \in I$

$$\sum_{k\in E_i}\left(p_k + \sum_{j\in U_k\setminus W_i} h_j\right) = \sum_{k\in E_i}\left(p_k + h_k + \sum_{j\in U_k} h_j\right) P\left\{I_{ki} \geq 0\right\}$$

- Proof based on echelon costs and similar to proof of single-item Newsvendor equation
- Relationship can be used as heuristic

Key transformations for divergent systems

Optimality equation can be rewritten as

$$\sum_{k\in E_j} \phi_k P\left\{I_{ki} \geq 0\right\} = \pi_j$$

$$P\{I_{ki} \geq 0\} = P\left\{q_j D_i + \sum_{s=1}^{K_k-1} Z_{sk}^{(j)} \leq q_j \Delta_j + \sum_{s=1}^{K_k-1} \tilde{\Delta}_{sk}^{(j)}, ((\ldots(Z_{K_k-1k}^{(j)} - \tilde{\Delta}_{K_k-1k}^{(j)})^+ + \ldots)^+ + Z_{1k}^{(j)} - \tilde{\Delta}_{1k}^{(j)})^+ = 0\right\}$$

$$\xi_j = \max_{k\in E_j} \sum_{s=1}^{K_k-1} \tilde{\Delta}_{sk}^{(j)}$$

$$x = q_j \Delta_j + \xi_j$$

$$P\{I_{ki} \geq 0\} = P\left\{q_j D_i + \sum_{s=1}^{K_k-1} Z_{sk}^{(j)} + \xi_j - \sum_{s=1}^{K_k-1} \tilde{\Delta}_{sk}^{(j)} \leq x, ((\ldots(Z_{K_k-1k}^{(j)} - \tilde{\Delta}_{K_k-1k}^{(j)})^+ + \ldots)^+ + Z_{1k}^{(j)} - \tilde{\Delta}_{1k}^{(j)})^+ = 0\right\}$$

Key transformations for divergent systems

$$P\left\{\theta = k\right\} = \phi_k$$

$$\sum_{k\in E_j} \phi_k P\left\{I_{ki} \geq 0\right\} = P\left\{I_{\theta i} \geq 0, \theta \in E_j\right\}$$

Optimality equation can be rewritten as

$$P\left\{q_j D_i + \sum_{s=1}^{K_\theta - 1} Z_{s\theta}^{(j)} + \xi_j - \sum_{s=1}^{K_\theta - 1} \tilde{\Delta}_{s\theta}^{(j)} \leq x, ((\ldots(Z_{K_\theta - 1\theta}^{(j)} - \tilde{\Delta}_{K_\theta - 1\theta}^{(j)})^+ + \ldots)^+ + Z_{1\theta}^{(j)} - \tilde{\Delta}_{1\theta}^{(j)})^+ = 0, \theta \in E_j\right\}$$

$$Y_j = \left(\sum_{s=1}^{K_\theta - 1} Z_{s\theta}^{(j)} - \sum_{s=1}^{K_\theta - 1} \tilde{\Delta}_{s\theta}^{(j)} + \xi_j \middle| ((\ldots(Z_{K_\theta - 2\theta}^{(j)} - \tilde{\Delta}_{K_\theta - 2\theta}^{(j)})^+ + \ldots)^+ + Z_{1\theta}^{(j)} - \tilde{\Delta}_{1\theta}^{(j)})^+ = 0, \theta \in E_j\right)$$

Optimality equation can be rewritten as

$$P\left\{q_j D_i + \left(Y_j \middle| Y_j \leq \xi_j\right) \leq x\right\} = \frac{\pi_j}{\sum_{n\in V_j} \pi_n}$$

Key transformations for divergent systems

Considering successor n of j, we find

$$\left(Y_j \middle| \theta \in E_n\right) = D_j + \xi_j - \Delta_n - \frac{\xi_n}{q_n} + \frac{1}{q_n}\left(Y_n \middle| Y_n \le \xi_n\right),$$

resulting in the following recursive relationship

$$\mathrm{P}\left\{Y_j \le x\right\} = \sum_{n\in V_j}\left(\frac{\sum_{k\in E_n}\phi_k}{\sum_{k\in E_j}\phi_k}\right) P\left\{D_j + \frac{1}{q_n}\left(Y_n \middle| Y_n \le \xi_n\right) \le x - \xi_j + \Delta_n + \frac{\xi_n}{q_n}\right\}$$

Computational efficiency

- Solving cost-balance equations or generalized Newsvendor equations is equivalent to solving a one-dimensional bisection scheme to find a target value in (0,1)
- Finding base stock levels for N-item divergent structure requires solution of N bisection schemes
  - As efficient as solving N single-echelon systems
- Finding base stock levels for N-item general systems requires solution of at most 3N bisection schemes

Transformation towards review period 1

- With nested review period the system runs through a "super cycle" with length equal to the review period of the root node R_N
- For each item we must distinguish R_N/R_i different cycles
- For the simplest (serial) situation we need to consider for item i R_N/R_i different random variables

$$W_{m_N, m_{N-1}, \ldots, m_i} = D\left[\sum_{j=i+1}^{N}\left(L_j + m_j R_{j-1}\right), \sum_{j=i}^{N}\left(L_j + m_j R_{j-1}\right)\right]$$

- Which are aggregated into

$$\hat{W}_i = D\left[\sum_{j=i+1}^{N}\left(L_j + M_j R_{j-1}\right), \sum_{j=i}^{N}\left(L_j + M_j R_{j-1}\right)\right], P\left\{M_j = m\right\} = \frac{R_{j-1}}{R_j}$$

Open issues to be resolved

- Recursive gamma fit approach becomes less accurate as the number of echelons increases
- The aggregation approach for the lot sizing case needs to take into account that as $\frac{R_{j-1}}{R_j}$ approaches infinity

$$\hat{W}_i = D\left(\sum_{j=i+1}^{N} \left(L_j + M_j R_{j-1}\right), \sum_{j=i}^{N} \left(L_j + M_j R_{j-1}\right) \right]$$

- approaches a uniform distribution, i.e. does not resemble a gamma distribution
- Currently this problem is addressed by not aggregating the random variables involved