Critical Queues and Random Graphs

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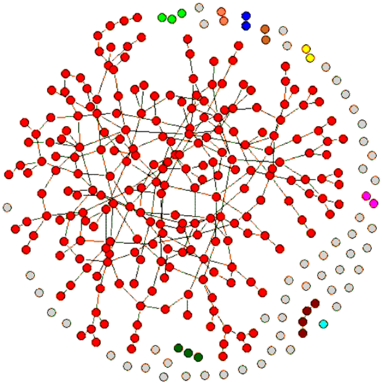
joint work with Shankar Bhamidi and Remco van der Hofstad

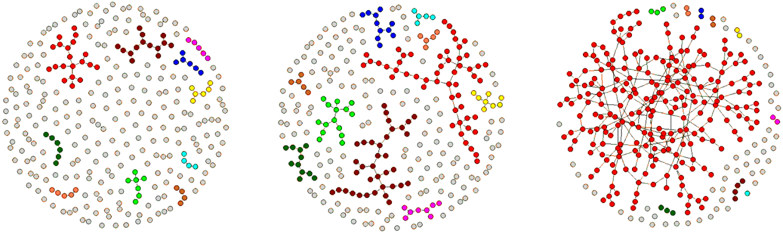
2nd Israeli-Dutch Workshop on QT

Erdős-Rényi random graph



Take n vertices labeled by $\{1, \ldots, n\}$ and put an edge between any pair independently with probability p

Phase transition (Erdős and Rényi 1960)



Consider $p = c/n$

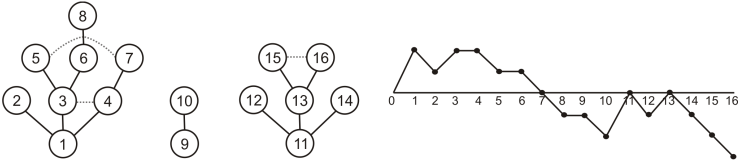
- For $c < 1$, the largest connected component has size $O(\log n)$
- For $c > 1$, the largest connected component has size $\Theta(n)$ (and the others are all $O(\log n)$)
- For $c = 1$, the largest component has size $\Theta(n^{2/3})$

Exploring the graph (Aldous 1997)

- Pick a vertex v not visited before and put it in a queue Q
- While Q is nonempty, pull a vertex v from the head of Q, draw edges to all the neighbors that have not been previously visited, and put these children at the back of Q

Label the vertices in their order of visitation, and define

$$Q(i) = Q(i-1) + \#\text{children of vertex } i-1$$



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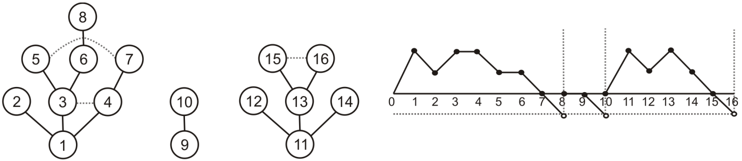


… a reflected random walk!

Exploring the graph (Aldous 1997)

$$\begin{aligned}\#\text{children of vertex } i &\overset{d}{=} \text{Binomial}(n - i - \#\text{queue} - 1, p)\\ &\overset{d}{\approx} \text{Binomial}(n, p)\\ &\overset{d}{\approx} \text{Poisson}(1)\end{aligned}$$

if we would ignore #queue, take $n \gg i$, and $p = 1/n$

However, the increments are not i.i.d., so we do *not* have a random walk, and the classical FCLT will not work. Luckily, the FCLT for martingales is suited for the dependencies we are facing.

Inhomogeneous random graphs

Poissonian graph process or *Norros-Reittu model* (2006):
Attach an edge with probability p_{ij} between vertices i and j, where

$$p_{ij} = 1 - \exp\left(-\frac{w_i w_j}{l_n}\right), \quad l_n = \sum_{i=1}^{n} w_i$$

Different edges are independent

The weight sequence $\mathbf{w} = (w_1, \ldots, w_n)$ is an i.i.d. sequence of random variables with distribution function F satisfying

$$\mathbb{E}[W^3] < \infty$$

Inhomogeneous random graphs

Equivalent to *random graphs with prescribed expected degrees*, studied by Chung and Lu (2002-2006); see also Bollobás, Janson and Riordan (2007)

$$p_{ij} = \min\left\{\frac{w_i w_j}{l_n}, 1\right\}$$

When $w_i = c$ we retrieve Erdős-Rényi with $p = c/n$

Also equivalent to *generalized random graphs* introduced by Britton, Deijfen and Martin-Löf (2005):

$$p_{ij} = \frac{w_i w_j}{l_n + w_i w_j}$$

See Janson (2010) for asymptotic equivalences of inhomogeneous random graphs

Where is the phase transition?

Define

$$\nu = \frac{\mathbb{E}[W^2]}{\mathbb{E}[W]}$$

Theorem (Bollobás-Janson-Riordan 2007)

- *largest component* $\sim \rho n$ *with* $\rho \in (0,1)$ *for* $\nu > 1$
- *largest component* $o(n)$ *for* $\nu < 1$

The phase transition occurs at $\nu = 1$

When the third moment exists

Let $\mu = \mathbb{E}[W], \sigma^2 = \mathbb{E}[W^3]/\mathbb{E}[W]$. Consider the process $(B_t^\beta)_{t\geq 0}$

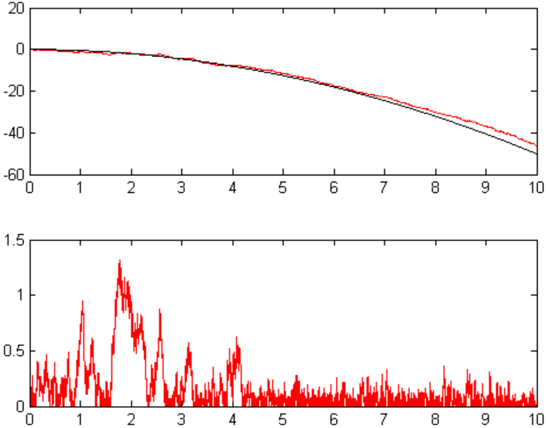
$$B_t^\beta = \sigma B_t + t\beta - t^2\sigma^2/(2\mu)$$

where $(B_t)_{t\geq 0}$ is standard Brownian motion. Define its reflected version as

$$R_t^\beta = B_t^\beta - \min_{0\leq u\leq t} B_u^\beta$$

Aldous (1997): Excursions of $(R_t^\beta)_{t\geq 0}$ can be ranked in increasing order as $\gamma_1(\beta) > \gamma_2(\beta) > \dots$

Reflected inhomogeneous Brownian motion



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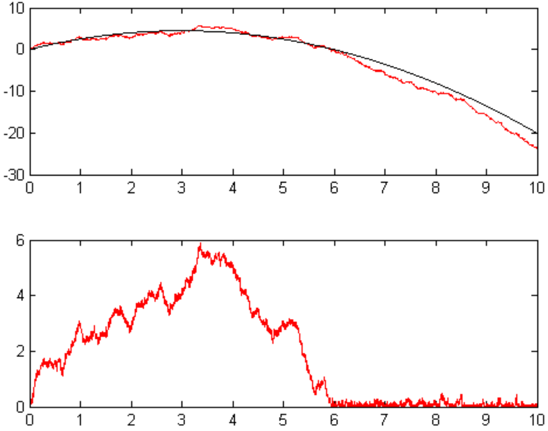

Theorem (Bhamidi-van der Hofstad-vL 2009)

Fix the Norros-Reittu graphs with weights

$$\tilde{w}_i = (1 + \beta n^{-1/3}) w_i$$

Assume that $\nu = 1$, *and* $\mathbb{E}[W^3] < \infty$. *Let* $|\mathcal{C}_{(1)}(\beta)| \geq |\mathcal{C}_{(2)}(\beta)| \ldots$ *denote sizes of the components in increasing order. Then, for all* $\beta \in \mathbb{R}$,

$$\left(n^{-2/3}|\mathcal{C}_{(i)}(\beta)|\right)_{i\geq 1} \xrightarrow{d} \left(\gamma_i(\beta)\right)_{i\geq 1}$$

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Alternative proof by Turova (2009). Proved by Aldous (1997) for $w_i = 1$ (Erdős-Rényi). Scaling limit studied in Groeneboom (1989), Martin-Löf (1998), Pittel (2001) and van der Hofstad, Janssen, vL (2010): connection with SIR model and e.g.

$$\mathbb{P}\left(|\mathcal{C}_{(1)}(\beta)| > tn^{2/3}\right) = \frac{4\sqrt{t}\exp\left(-\frac{1}{8}t(t-2\beta)^2\right)\left(1 + O(t^{-3/2})\right)}{\sqrt{2\pi}(t-2\beta)(3t-2\beta)}$$

When the third moment does not exist...

Take

$$w_i = [1 - F]^{-1}(i/n)$$

where $F(x)$ a distribution function with $1 - F(x) \sim cx^{-(\tau-1)}$

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where $F(x)$ a distribution function with $1 - F(x) \sim cx^{-(\tau-1)}$
Simple example:

$$F(x) = \begin{cases} 0 & \text{for } x < a \\ 1 - (a/x)^{\tau-1} & \text{for } x \geq a \end{cases}$$

so $[1 - F]^{-1}(u) = a(1/u)^{-1/(\tau-1)}$ and $w_i = a(n/i)^{1/(\tau-1)}$
Also,

$$\mathbb{E}[W] = \frac{a(\tau-1)}{\tau-2} \qquad \mathbb{E}[W^2] = \frac{a^2(\tau-1)}{\tau-3}$$

so that critical case arises when

$$\nu = \frac{\mathbb{E}[W^2]}{\mathbb{E}[W]} = \frac{a(\tau-2)}{\tau-3} = 1 \quad \Longleftrightarrow \quad a = \frac{\tau-3}{\tau-2}$$

Theorem (Bhamidi-van der Hofstad-vL 2009)

Fix the Norros-Reittu graphs with weights

$$\tilde{w}_i = (1 + \beta n^{-(\tau-3)/(\tau-1)}) w_i$$

Assume that $\nu = 1$ *and* $\tau \in (3, 4)$. *Let* $|\mathcal{C}_{(1)}(\beta)| \geq |\mathcal{C}_{(2)}(\beta)| \dots$ *denote sizes of components arranged in increasing order. Then,*

$$\left(n^{-(\tau-2)/(\tau-1)} |\mathcal{C}_{(i)}(\beta)|\right)_{i \geq 1} \xrightarrow{d} \left(H_i(\beta)\right)_{i \geq 1}$$

with $H_i(\beta)$ *corresponding to ordered hitting times of 0 of a certain fascinating 'thinned' Lévy process*

Thinned Lévy process

$$\mathcal{S}_t = c + bt + \sum_{i=2}^{\infty} i^{-a}\left[\mathcal{I}_i(t) - ti^{-a}\right]$$

with $\mathcal{I}_i(t) = \mathbf{1}_{\{\mathrm{Exp}(i^{-a})\in[0,t]\}}$

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Compare with the spectrally positive Lévy process

$$\mathcal{R}_t = c + bt + \sum_{i=2}^{\infty} i^{-a}\left[N_i(t) - ti^{-a}\right]$$

with (N_i) independent Poisson processes with rates i^{-a}

$$\mathcal{S}_t \leq \mathcal{R}_t$$

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$\mathcal{R}_t$ is a poor approximation for $\mathcal{S}_t$ (thinning is important)

Special case of the multiplicative coalescents in Aldous and Limic (1997,1998). Detailed study with Elie Aidekon, Remco van der Hofstad and Sandra Kliem.

Proof: weak convergence stochastic processes

(1) Exploration of components

(2) Removal of possible further neighbors due to their exploration: *depletion of points effect*

(3) Under the right scaling, the exploration process weakly converges. Cluster sizes correspond to excursion lengths limiting process having an *increasing negative drift*

- $\mathbb{E}[W^3] < \infty$: exploration process has finite variance steps, so that Brownian motion appears in limit, and

$$\mathbb{P}(1 \in \mathcal{C}_{\max}) \to 0 \quad \text{(power to the masses)}$$

- $\mathbb{E}[W^3] = \infty$: exploration process is dominated by vertices with high weights, and

$$\mathbb{P}(1 \in \mathcal{C}_{\max}) \to p(\beta) \in (0, 1) \quad \text{(power to the wealthy)}$$

References

[1] Aldous, D. (1997) Brownian excursions, critical random graphs and the multiplicative coalescent. *AoP* **25**, 812–854.

[2] Bollobás, B., Janson, S. and Riordan, O. (2007) The phase transition in inhomogeneous random graphs. *RSA*, **31**, 3–122.

[3] Bhamidi, S., van der Hofstad, R. and van Leeuwaarden, J. (2009) Scaling limits for critical inhomogeneous random graphs with finite third moments. `http://arxiv.org/abs/0907.4279`. To appear in *EJP*.

[4] Bhamidi, S., van der Hofstad, R. and van Leeuwaarden, J. (2009) Novel scaling limits for critical inhomogeneous random graphs. `http://arxiv.org/abs/0909.1472`.