Some issues for aggregate models of production systems

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Collaborators

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- **Karl Kempf**, Intel Corporation

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Outline

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1. Aggregate Simulations
2. Control of aggregate models

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3. Transient Clearing Functions
4. Continuous models for production lines with finite buffers

Aggregate Simulations

Usual model: Faithful representation of the factory using *Discrete Event Simulations*, e.g. χ (TU Eindhoven)

Problem:

Simulation of production flows with stochastic demand and stochastic production processes requires Monte Carlo Simulations

Takes too long for a decision tool

[1]Dieter Armbruster, Daniel Marthaler, Christian Ringhofer, Karl Kempf, Tae- Chang Jo: Operations Research **54**(5), 933 -950, 2006

Fundamental Idea:
Model high volume, many stages, production via a fluid.

Basic variable

product density (mass density) $\rho(\mathbf{x}, \mathbf{t})$.
x- is the position in the production process, $x \in [0, 1]$.
- degree of completion
- stage of production

Mass conservation and state equation

$$\frac{\partial \rho}{\partial t} + \frac{\partial F}{\partial x} = 0$$
$$F = \rho v_{eq}$$

Typical models for the equilibrium velocity v_{eq} (state equation) are

$$v_{eq}^{traffic}(\rho) = v_0\left(1 - \frac{\rho}{\rho_c}\right)$$
$$v_{eq}^{Q} = \frac{\mu}{1 + L}$$
$$v_{eq} = \Phi(L)$$

with L the total load (WIP) given as $L(\rho) = \int_0^1 \rho(x,t)dx$
Note: $\Phi(L)$ may be determined experimentally or theoretically

Equivalence

A clearing function $\Omega(L)$ giving the outflux as a function of WIP in steady state is completely equivalent to v_{eq}.

$$\Omega(L) = Lv_{eq}$$

e.g.

$$\begin{aligned} \Omega_Q &= \frac{\mu L}{1+L} \\ \Omega_K &= \alpha L(1-e^{-\beta L}) \end{aligned}$$

Validation: DES vs. fluid simulation

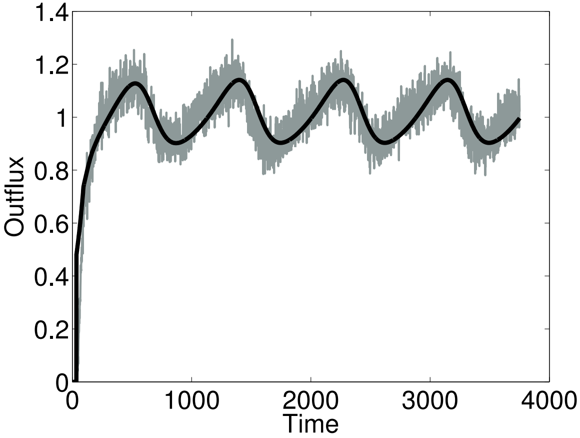


Figure: Throughput as a function of time for a sinusoidally varying input

Control of aggregate models

Tracking for a continuum model

Model: PDE model based on a product density $\rho(x,t)$ and a state equation for the velocity.

$$\begin{aligned}
\rho_t(x,t) + v_{eq}^Q \rho_x(x,t) &= 0, \quad (x,t) \in [0,1] \times [0,\infty) \\
\rho(x,0) &= \rho_0(x), \quad x \in [0,1] \\
v_{eq}^Q &= \frac{v_{max}}{1+L} \\
\lambda(t) &= v(\rho)\rho(x,t)|_{x=0}
\end{aligned}$$

where $\lambda(t)$ is the influx.

Problem setup

- a fixed end time $\tau > 0$.
- an initial profile $\rho_0(x)$.
- $d(t)$ - the demand at time t. $d(t) \in L^2([0, \tau])$.

Find the influx $\lambda(t), t \in [0, \tau]$:s.t.

$$j(\rho, \lambda) = \frac{1}{2} \int_0^{\tau} \left(v_{eq}^{Q}(\rho)\rho(1, t) - d(t) \right)^2 dt$$

is minimal

Method

- variational approach via adjoint calculus
- leads to two coupled PDEs
- solving allows to determine $\frac{dj(\lambda)}{d\lambda}$
- gradient search algorithm finds a local minimum

System Reactivity

The speed

$$v = \frac{v_{max}}{1 + \int_0^1 \rho(s,t)\, ds}$$

depends on v_{max}.

Sinusoidal demands with:

- $v_{max} = 1$, inert factory
- $v_{max} = 3$, agile factory

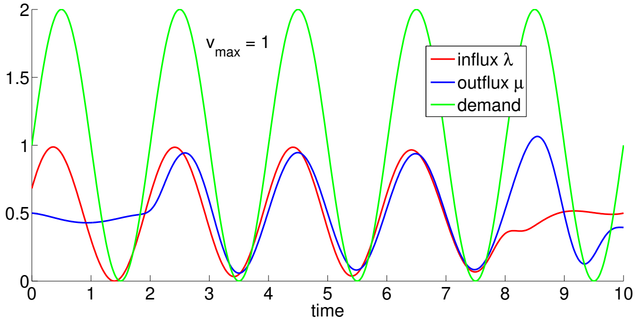


Figure: v_{max} of 1

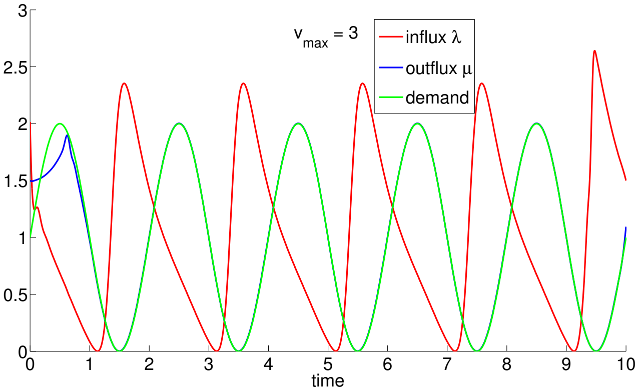


Figure: v_{max} of 3

Transient Clearing Functions[2]

[2] Details see poster by Jasper Fontejin

Modeling transient behavior through clearing functions

Recall H. Missbauer's talk

- Consider an $M/M/1$ queue for a fixed time interval $[t, t+\tau]$ and an arrival rate $\lambda(t)$.
- Clearing function:

$$\text{Expected output}(t) = \Phi(\text{Expected load L})(t)$$

with

$$L(t) = w + A = w + \int_t^{t+\tau} \lambda(s)ds$$

w is the initial WIP.

- Result: Φ depends on w, the arrival A and variance of the initial WIP σ_w^2.

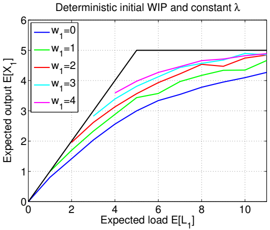
It is more complicated

Functional form of λ

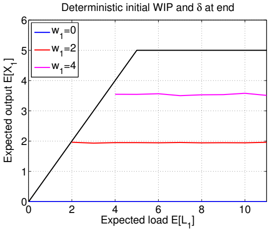
Choose initial WIP, arrival A and vary $\lambda(t)$

- Choose $\lambda(t) = c$
- linearly increasing inter arrival rate
- linearly decreasing inter arrival rate
- impulse at $s = 0$
- impulse at $s = \tau$

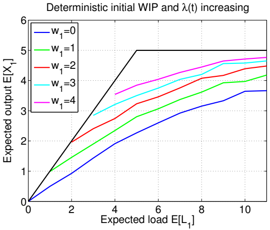
Results



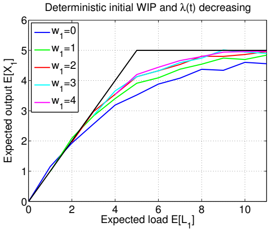
(a) constant λ



(b) impulse at end



(c) λ increasing

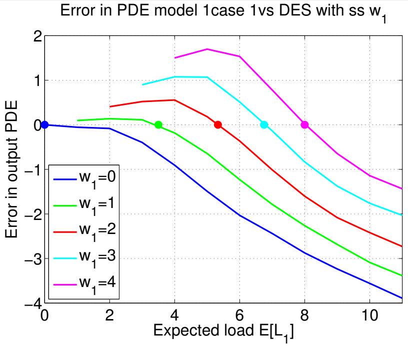


(d) λ decreasing

How good is the PDE model for those transient cases?

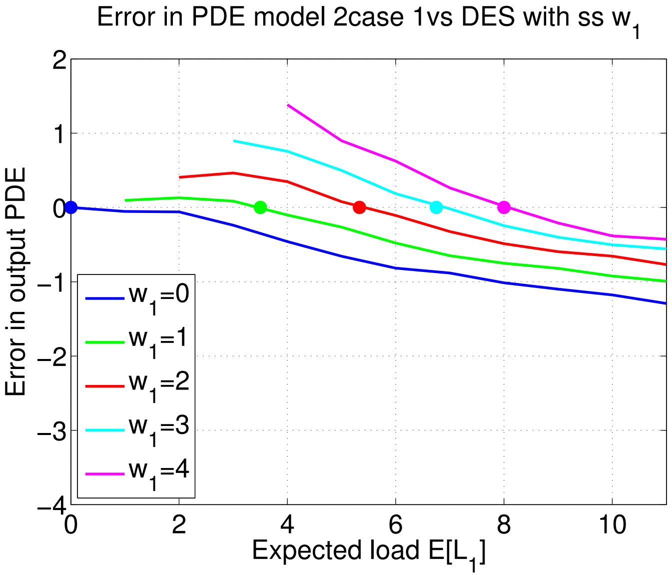
PDE model is adiabatic

Should be good for slow changing influx.



Modeling the time evolution of the velocity

$$
\begin{aligned}
\frac{\partial \rho}{\partial t} + \frac{\partial v \rho}{\partial x} &= 0 \\
\frac{\partial v}{\partial t} + v \frac{\partial v}{\partial x} &= 0 \\
\rho v|_{x=0}(t) &= \lambda(t) \\
v(0,t) &= v_{eq}(t)
\end{aligned}
$$


Error in PDE model 2case 1vs DES with ss w1
Error in output PDE
Expected load E[L1]
2
1
0
−1
−2
−3
−4
0
2
4
6
8
10
w1=0
w1=1
w1=2
w1=3
w1=4

Continuous models for production lines with finite buffers

Production lines with finite buffers

Current assumption

Buffers can become infinite
ρ can have δ-measures
flux may be restricted but not density

Production lines for larger items, e.g. cars

There exists only a small buffer between machines
Need to implement a limit on ρ in our model

Experiment

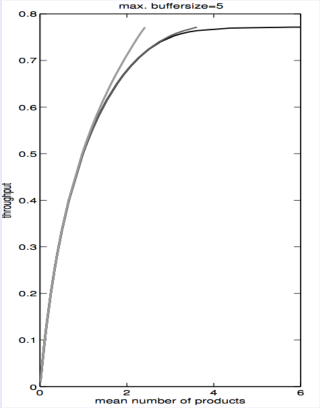
100 identical machines with capacity $\mu = 1$
all buffers between machines have identical capacity of M

1. fill an empty factory with a constant influx rate $\lambda < 1$
2. shut down the last machine
3. factory fills up and stops working when the first buffer is at its maximum.
4. restart last machine and drain the factory until it reaches steady state again.

[3] P. Goossens, Modeling of manufacturing systems with finite buffer sizes using PDEs, Masters Thesis, TU Eindhoven, 2007

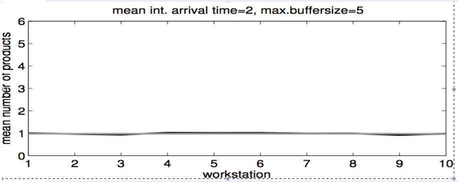
Model needs to explain:

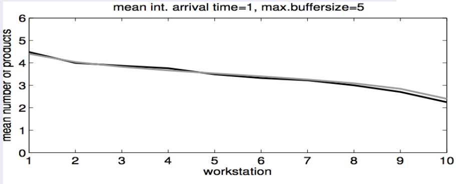
- The maximal steady state throughput λ_{max} of the production line is much lower than 1



More:

- The steady–state WIP distribution $\rho_{ss}(x)$ for $\lambda << 1$ is constant in x
- The steady–state WIP distribution $\rho_{ss}(x)$ for $\lambda \approx \lambda_{max}$ decays almost linearly in x





More:

- At shut down, the production line is filled up by a backwards moving wave.
  wave speed is

$$v_{shutdown} = \frac{\lambda}{M - \int_0^1 \rho_{ss}(x)dx}. \tag{1}$$

- The transient drain depends on the influx λ.
  - If $\lambda \approx \lambda_{max}$ then the factory drains from the end.
  - If $\lambda < \lambda_{max}$ then WIP is reduced by a wave "eating" into it from upstream and at the same time WIP uniformly drains downstream.

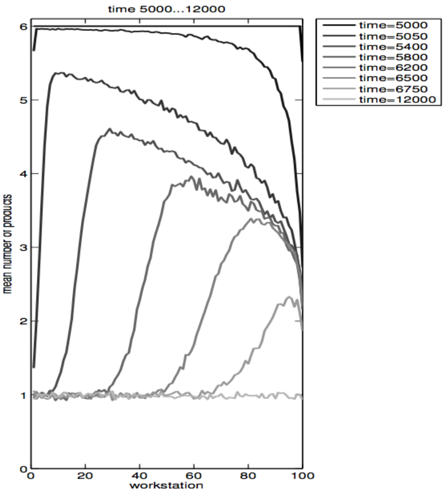


Figure: $\lambda < \lambda_{max}$. The WIP distribution drifts downwards and "gets eaten" from the back

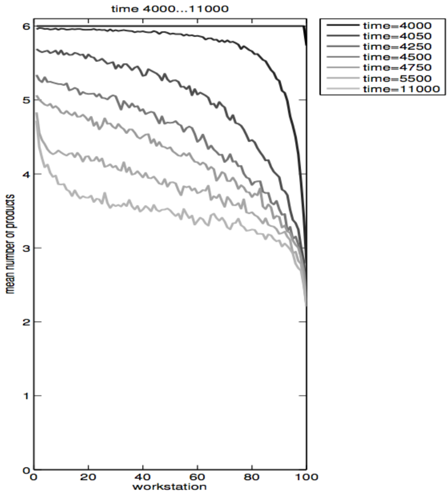


Figure: $\lambda \approx \lambda_{max}$, the system approaches the steady state distribution almost uniformly in space.

Two fundamental stochastic processes

- The production process with mean processing rate $\mu = 1$.
- The blocking process when the buffer becomes full.

Together they lead to an inhomogeneous processing rate

$$\tilde{\mu} = c(x)\mu.$$

We make three assumptions for $c(x)$;

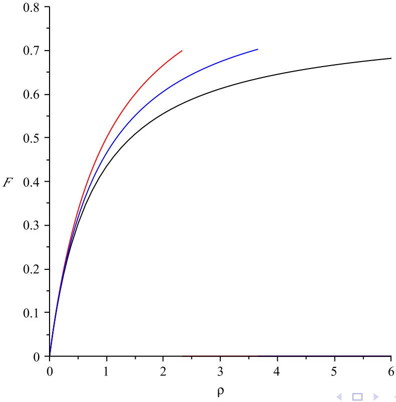
- $c(1) = 1$.
- $c(x)$ linearly increases with the steady state influx λ.
- $c(x)$ linearly increases as a function of x.

Consistent Assumption:

$$\tilde{\mu} = c(x)\mu = \lambda k(x-1) + \mu$$

Inhomogeneous and discontinuous flux

$$F(\rho, x) := \begin{cases} \frac{\mu\rho}{1+\rho+k\rho(1-x)} & \text{for } \rho < M \\ 0 & \text{for } \rho \geq M. \end{cases} \tag{2}$$



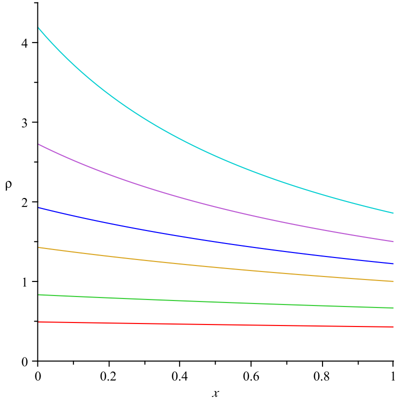
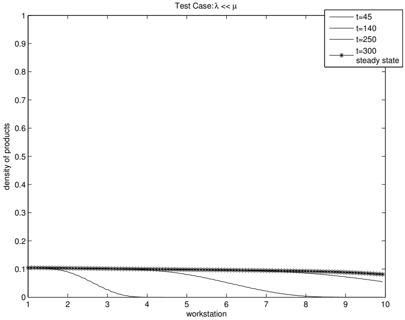


Figure: Steady states for a flux function (2) and different values for the inflow densities λ

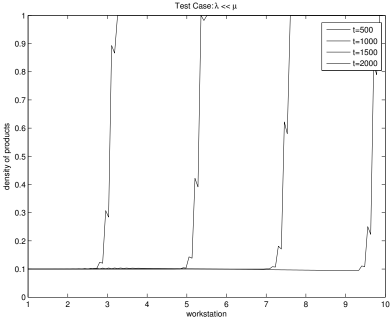
Riemann problem

For different initial conditions we get different kinetic waves:

- a rarefaction - speed $\lambda = f'(\rho)$. Filling wave - start at a traffic light.
- a shock wave - speed $s = \frac{f(\rho_l)}{\rho_l - M}$. Blocking wave.
- a shock wave traveling with infinite speed. Information wave after restart.
- This wave is followed by a classical rarefaction wave emanating at $x = 1$ and a shock wave emanating at $x = 0$

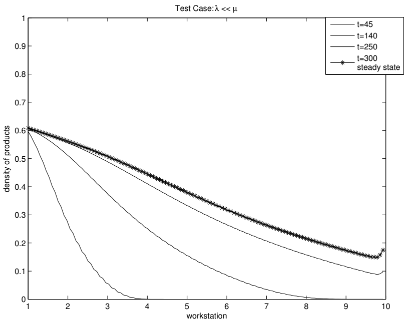


(a) filling an empty factory

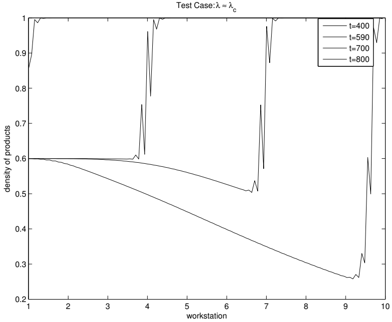


(b) blocking

Figure: $\lambda < \lambda_{max}$.



(a) filling an empty factory



(b) blocking

Figure: $\lambda \approx \lambda_{max}$.