The Variance of Production Counts over a Long Time Horizon

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Contains joint work with

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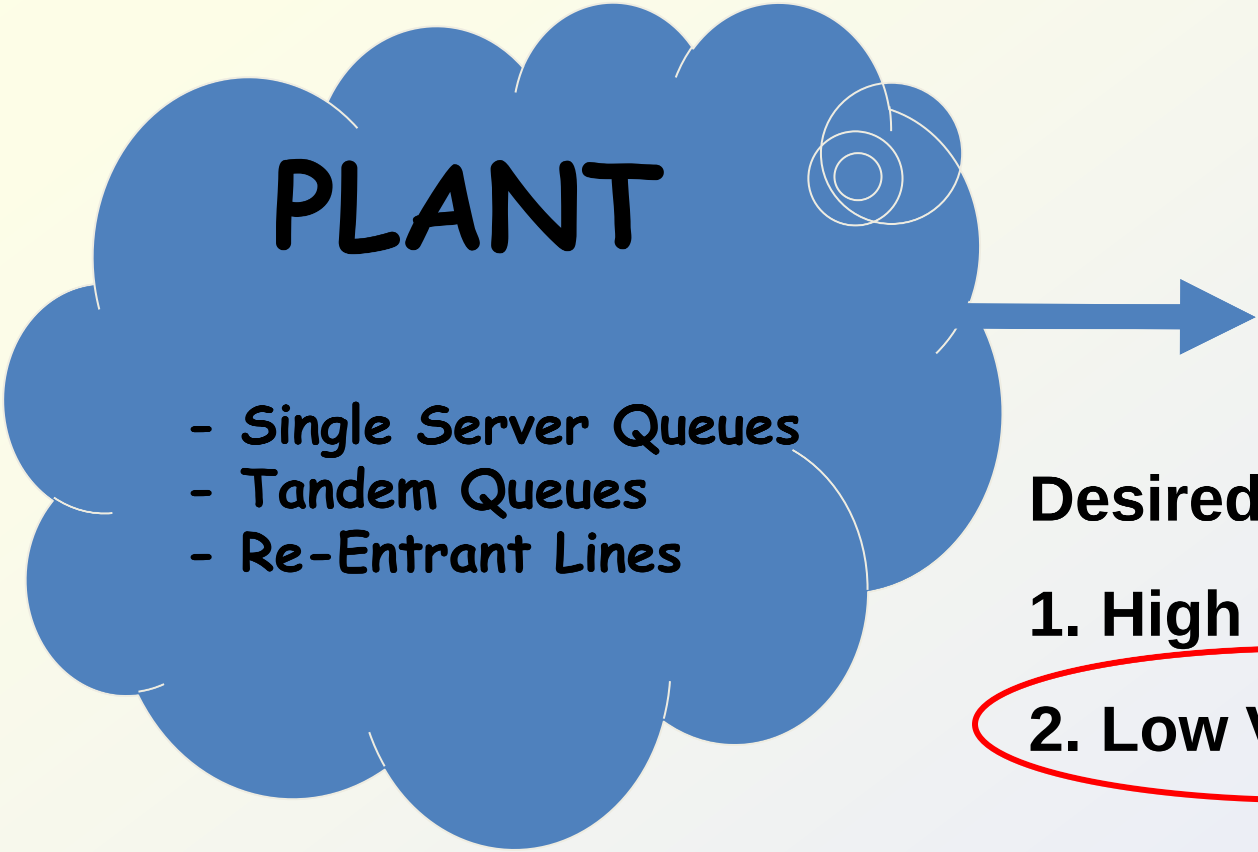
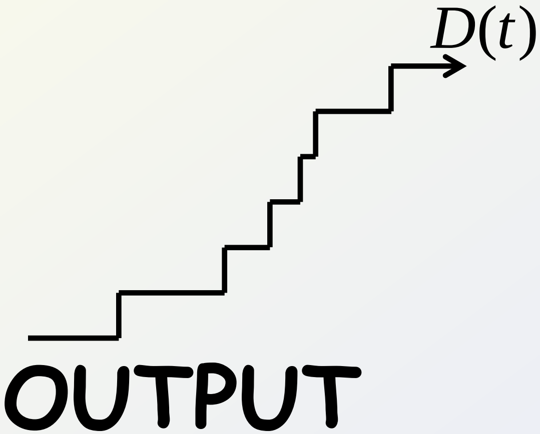
Problem Domain: Queueing Output Processes





Desired over long term:

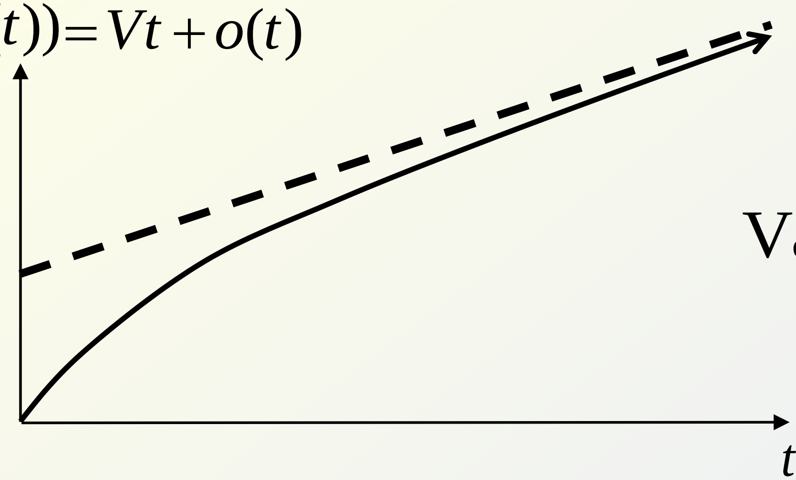
1. High Throughput

2. Low Variability

Our focus: $Var\left(D(T)\right)$ **for large T**

Variance Curves

$$\mathrm{Var}(D(t)) = \bar{V}t + o(t)$$



$$\mathrm{Var}\left(D(T)\right) \approx \bar{V}T$$

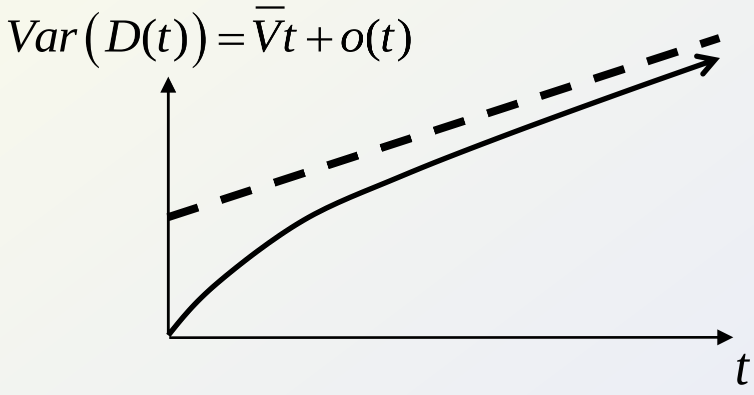
Example: Stationary stable M/M/1, D(t) is PoissonProcess(λ): $Var\left(D(t)\right) = \lambda t$

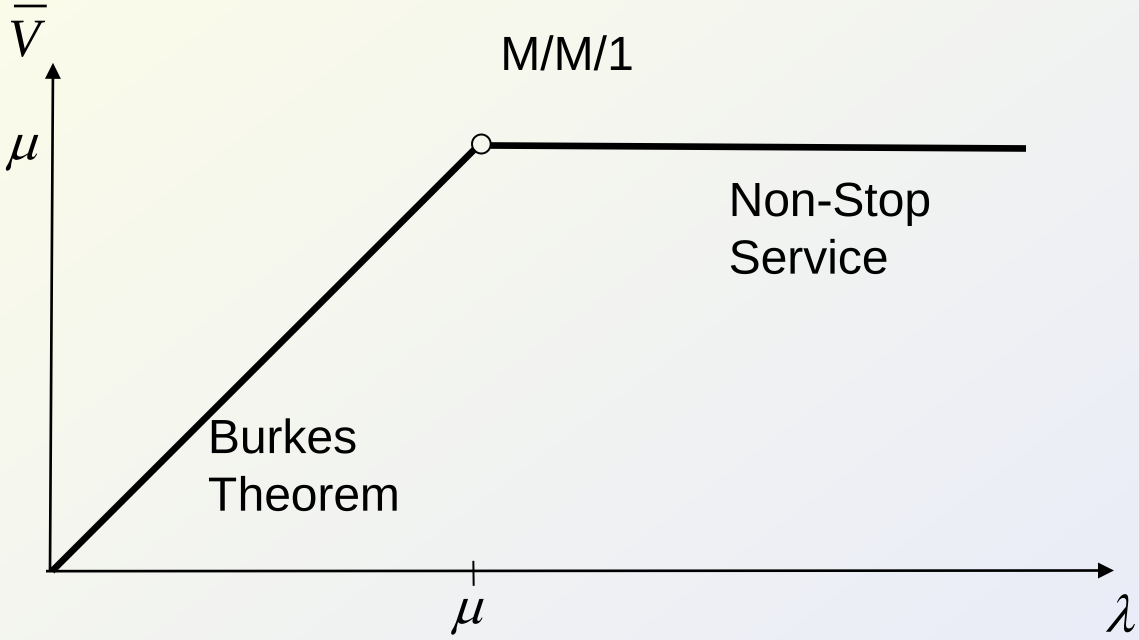
Example: Stationary M/M/1/1 with $\lambda = \mu$
D(t) is RenewalProcess(Erlang(2,λ)): $Var\left(D(t)\right) = \frac{1}{4}\lambda t + \frac{1}{8} - \frac{1}{8}e^{-2\lambda t}$

$\bar{V} \equiv$ **Asymptotic Variance Rate of Outputs**

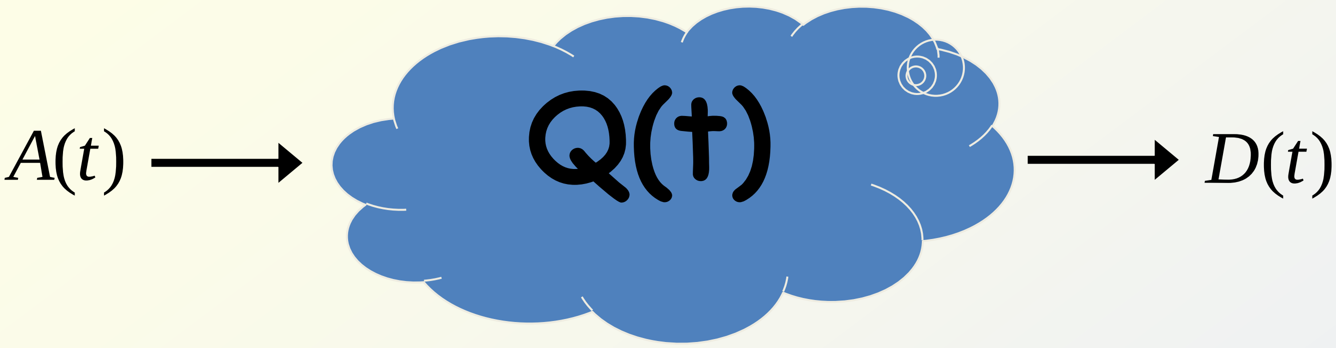
For Renewal Processes: $\bar{V} = \frac{\sigma^2}{m^3} = \frac{1}{m}c^2$

Asymptotic Variance Rate





The Basic Loss-Less **Stable** Queueing System



$$Q(t) = Q(0) + A(t) - D(t)$$

$$\frac{Var\left(D(t)\right)}{t} = \frac{Var\left(A(t)\right)}{t} + \frac{Var\left(Q(t)\right)}{t} - 2\frac{Cov\left(A(t),Q(t)\right)}{t}$$

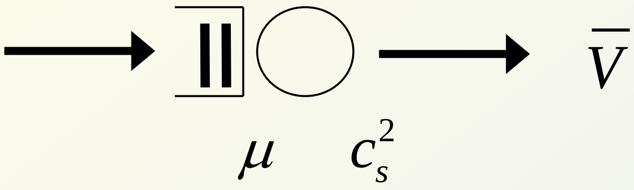
$$\boxed{\overline{V} = \overline{V}_A}$$

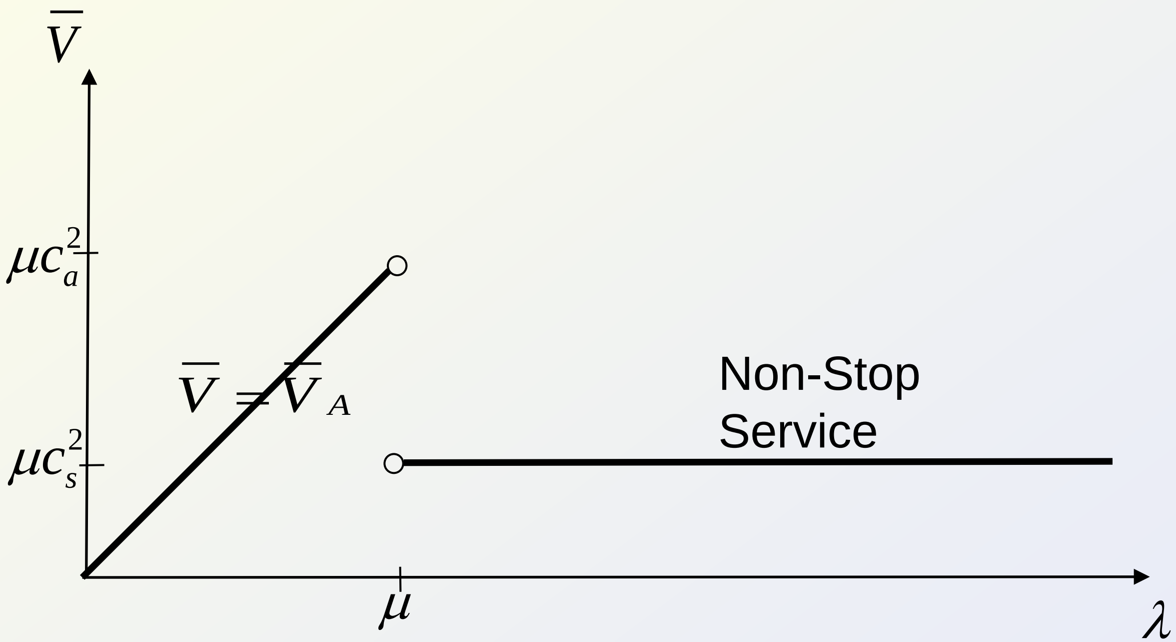
Our main focus:

Overloaded and **critically loaded** systems

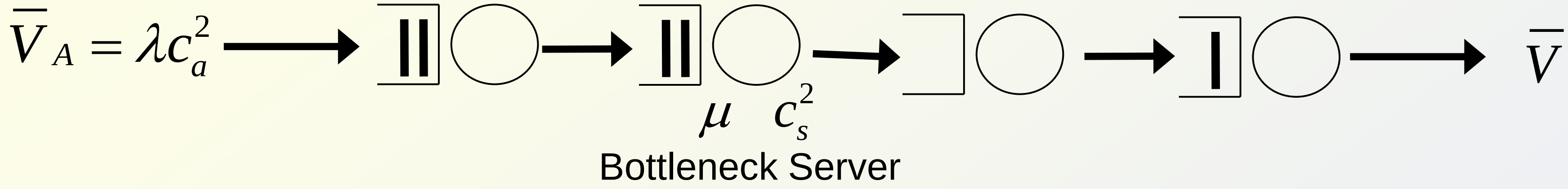
GI/G/1

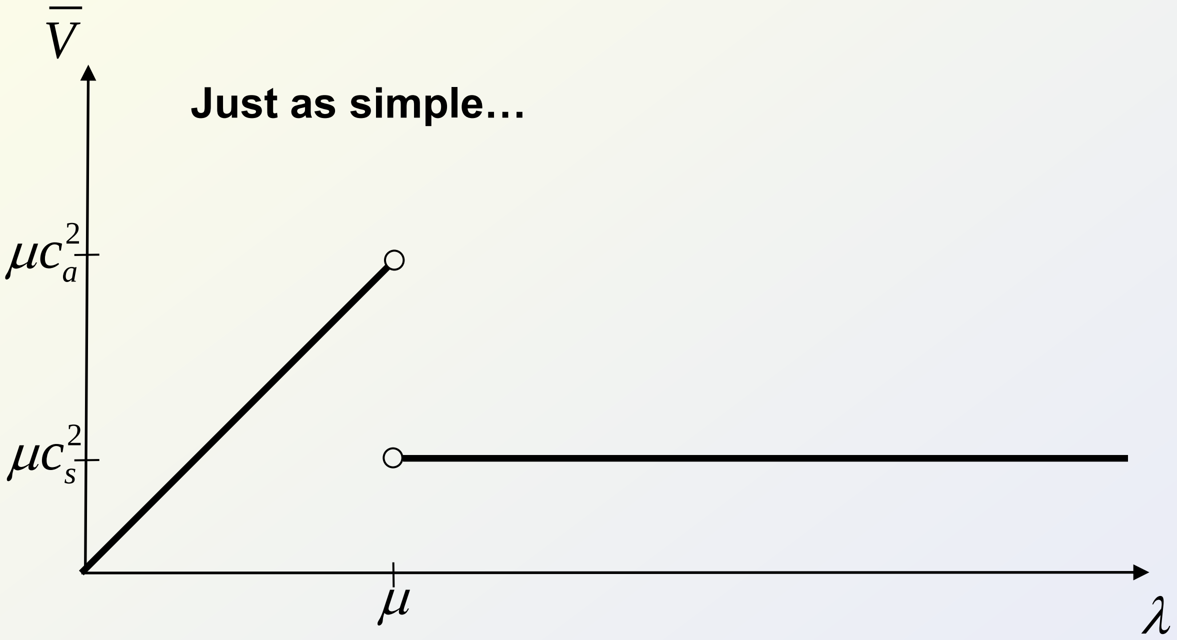
$\bar{V}_A = \lambda c_a^2$



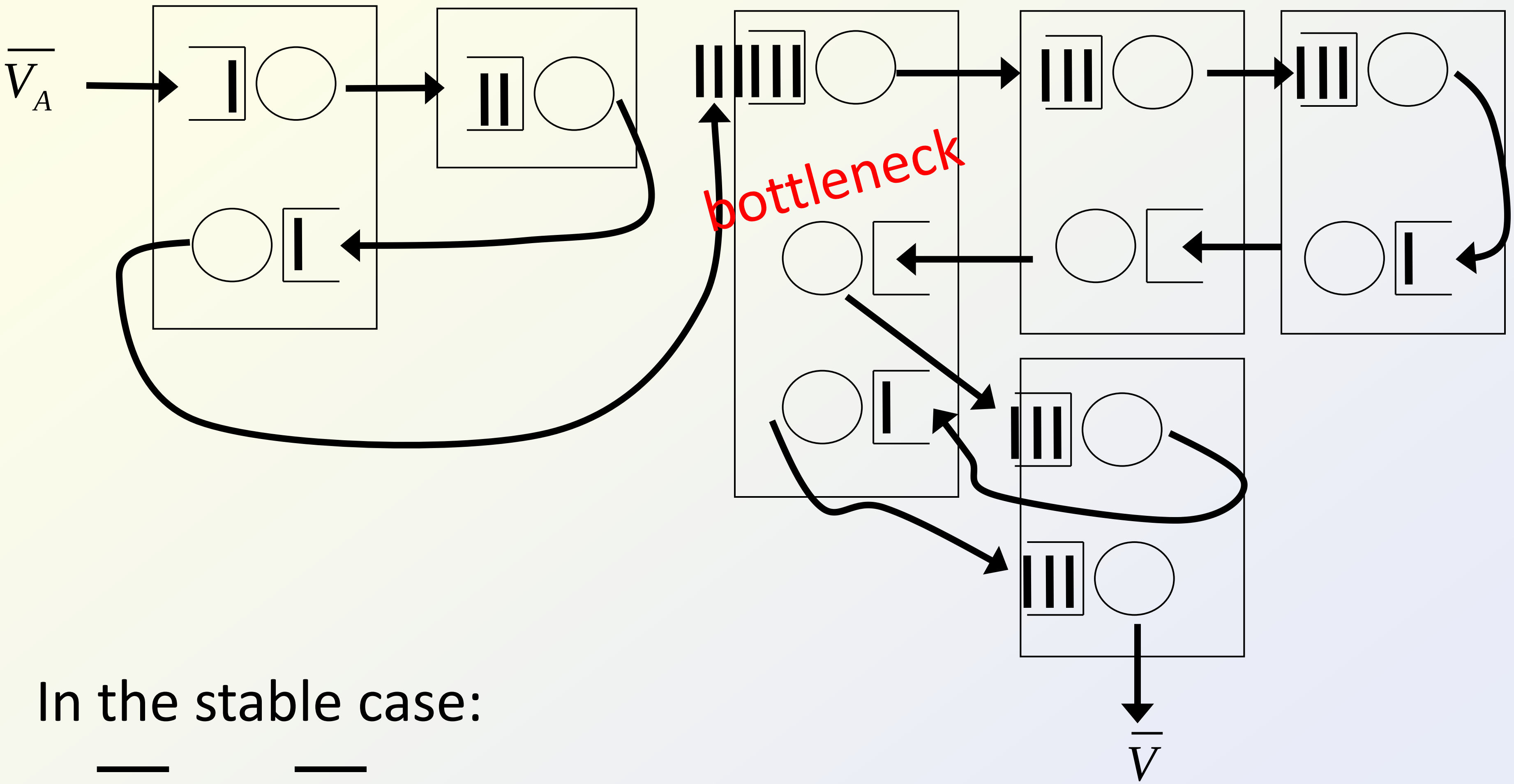


Queues in Tandem (with 1 bottleneck)





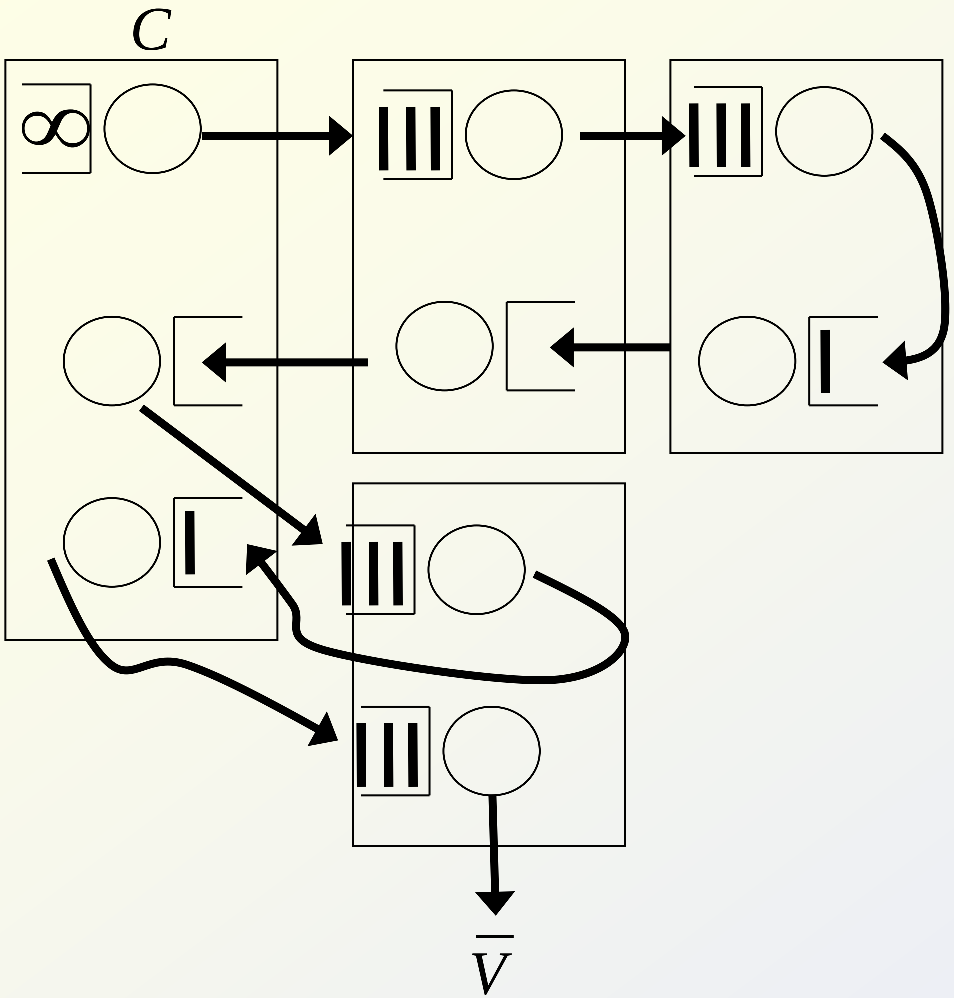
Re-entrant Line



In the stable case:

$$\overline{V} = \overline{V}_A$$

Overloaded case --> **Infinite Supply Re-entrant Line**



Model: Means: $m_1, \ldots, m_k$

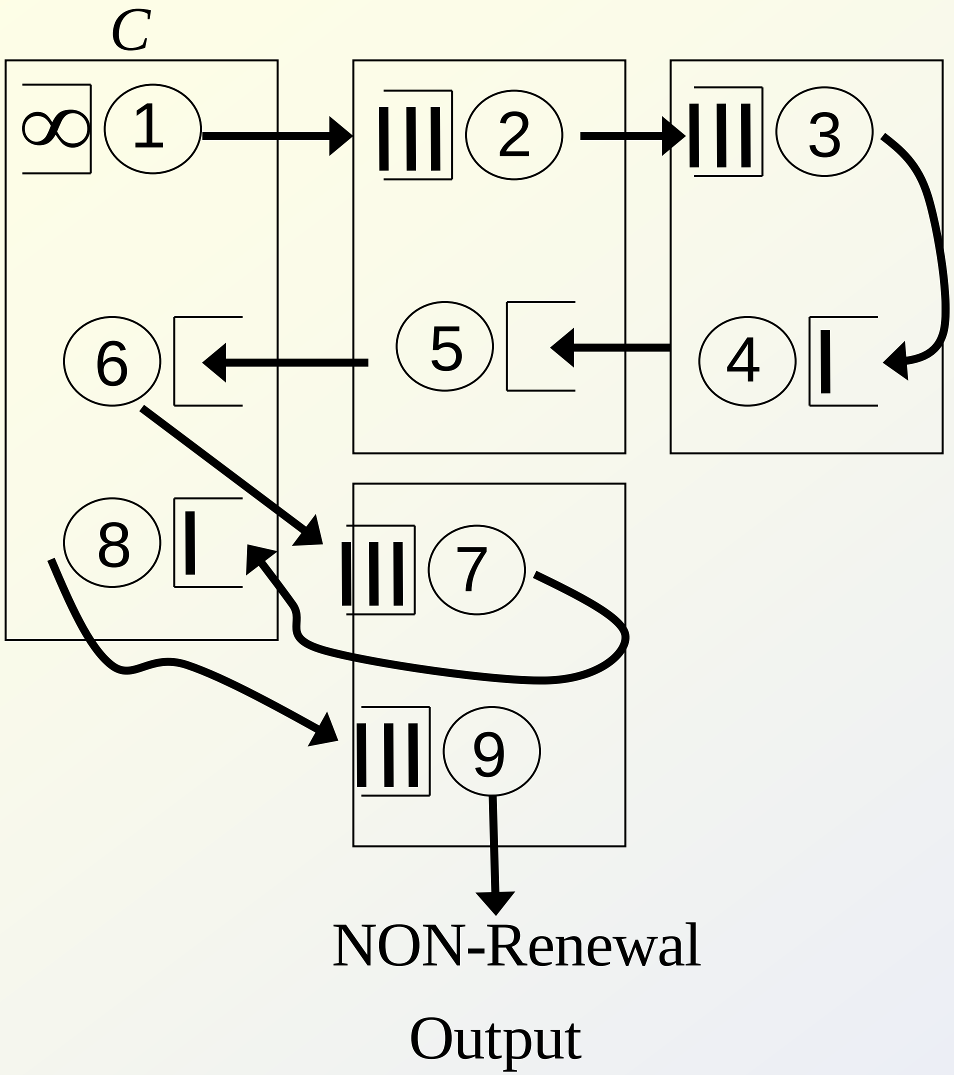
Variances: $\sigma_1^{\ 2}, \ldots, \sigma_k^{\ 2}$

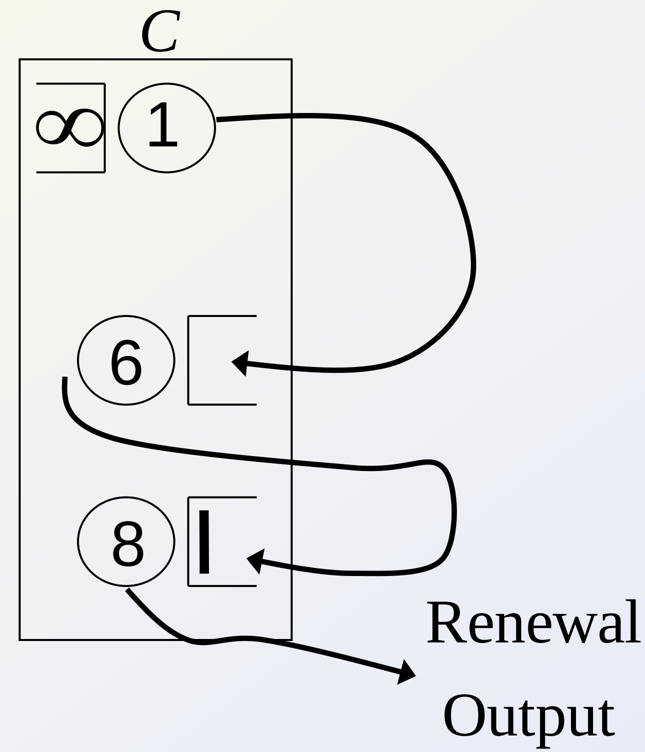
C is bottleneck

Control: Stable

Result: $$\bar{V} = \frac{\sum_{k \in C} \sigma_k^{\ 2}}{\left(\sum_{k \in C} m_k \right)^3}$$

Overloaded case --> **Infinite Supply Re-entrant Line**





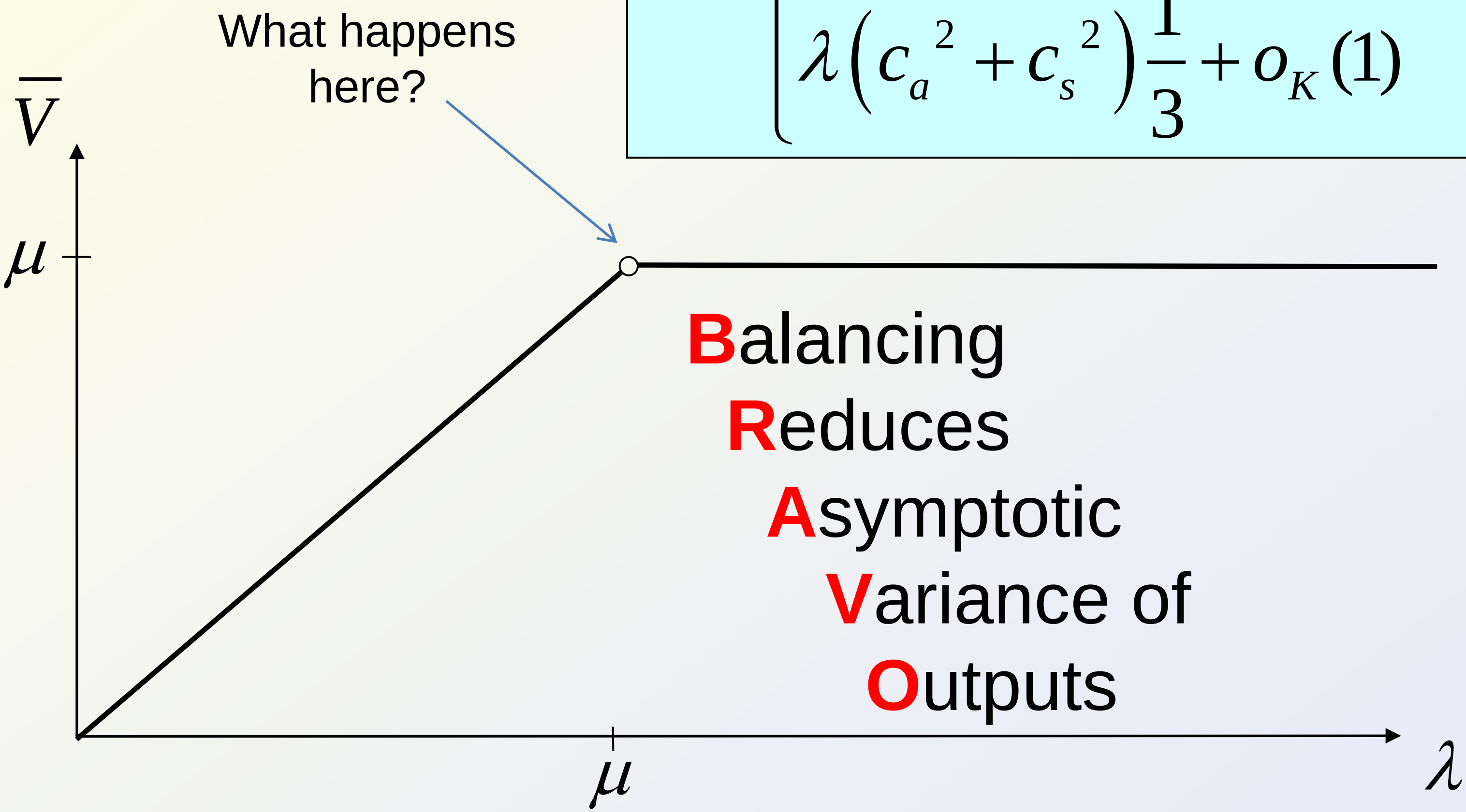
Result: $$\overline{V} = \frac{\sum_{k\in C} \sigma_k^{\,2}}{\left(\sum_{k\in C} m_k\right)^3}$$

Shocking result* coming up…

* at least for me

Back to Single Server (GI/G/1/K)

$$\overline{V} = \begin{cases} \lambda\left(c_a{}^2 + c_s{}^2\right)\left(1 - \dfrac{2}{\pi}\right) & K = \infty \\ \lambda\left(c_a{}^2 + c_s{}^2\right)\dfrac{1}{3} + o_K(1) & K < \infty \end{cases}$$

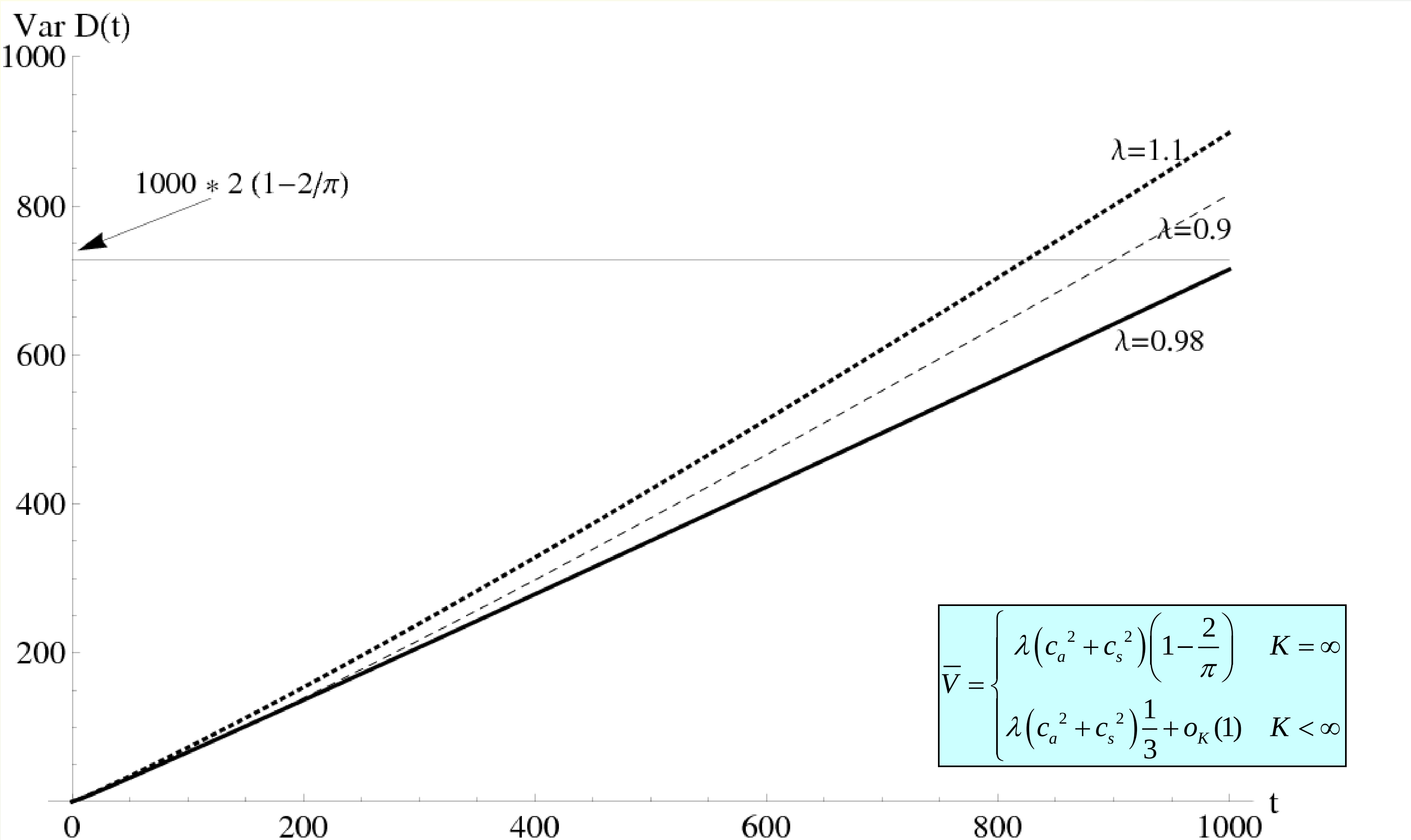


Balancing
Reduces
Asymptotic
Variance of
Outputs

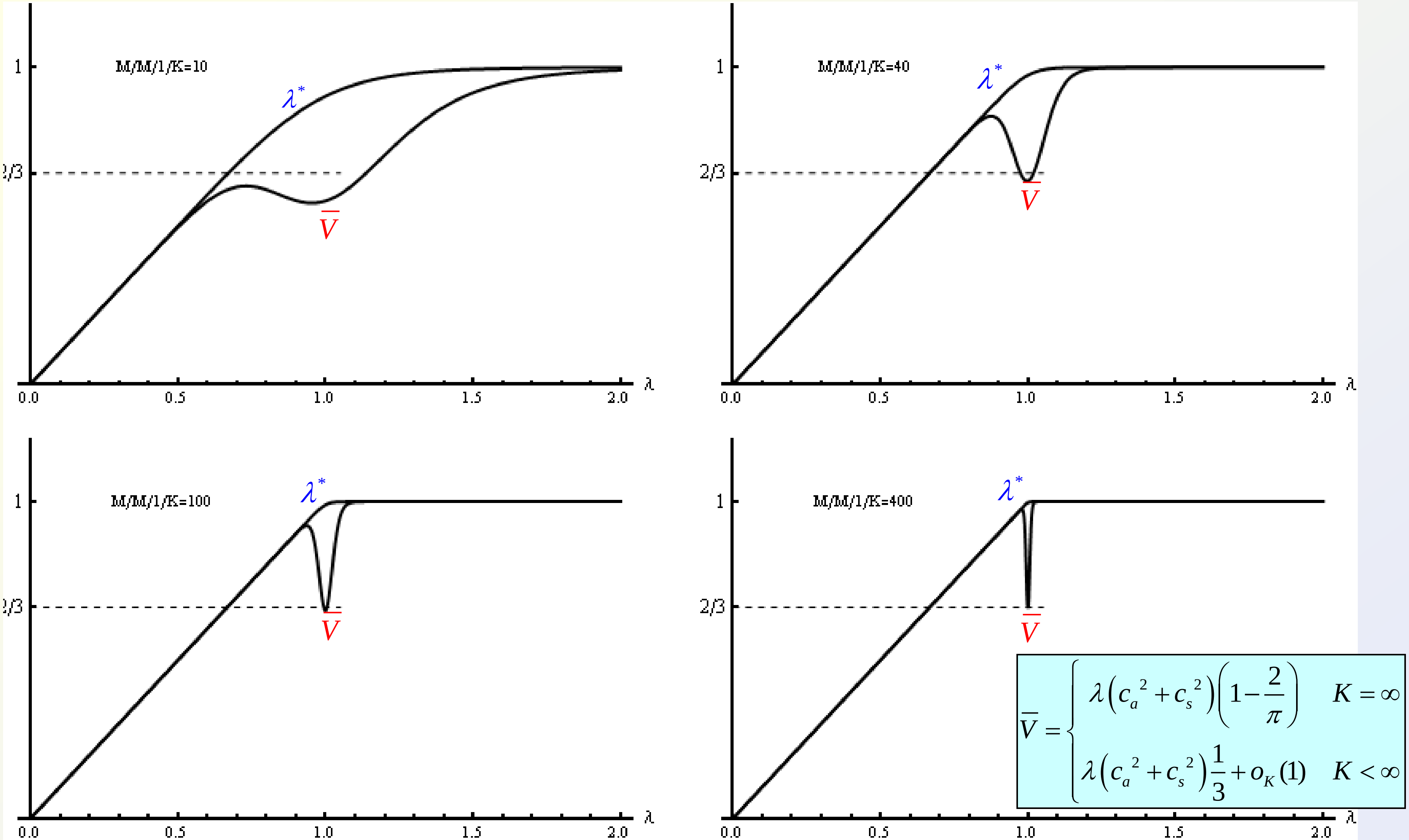
Note: the figure assumes $c_a{}^2 = c_s{}^2$

BRAVO Effect (illustration for M/M/1)

More than a singular theoretic phenomenon



BRAVO Effect (for M/M/1/K)

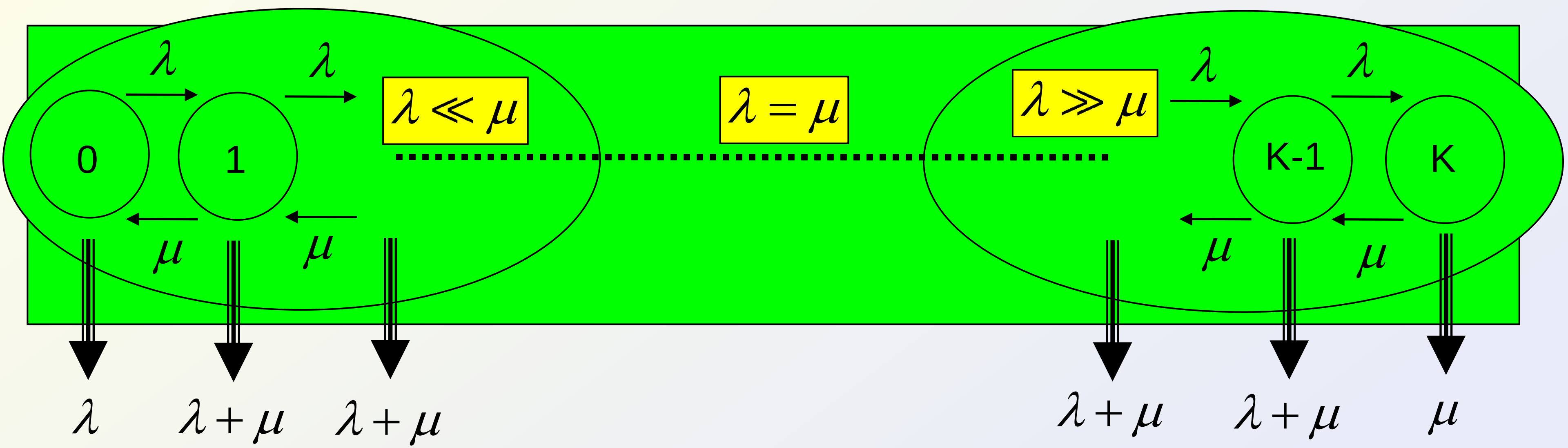


$$\overline{V} = \begin{cases} \lambda\left(c_a{}^2 + c_s{}^2\right)\left(1 - \dfrac{2}{\pi}\right) & K = \infty \\ \lambda\left(c_a{}^2 + c_s{}^2\right)\dfrac{1}{3} + o_K(1) & K < \infty \end{cases}$$

Some (partial) intuition for M/M/1/K

$\overline{V}_M \equiv$ Asymptotic variance of number of transitions

Easy to see: $\overline{V} = 4\overline{V}_M$



Questions?