The Contraction Method on $\mathcal{C}([0,1])$ and Donsker's Theorem

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joint work with Ralph Neininger

Quickselect / Find

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- Stop, if the considered set is of size 1.

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Scaling gives

$$Y_n := \frac{X_n}{n} \stackrel{d}{=} \frac{I_n}{n} Y_{I_n} + \frac{n-1}{n}$$

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Observe: $\mu \in \mathcal{M}(\mathbb{R})$ satisfies (1), iff $F(\mu) = \mu$ for

$$F : \mathcal{M}(\mathbb{R}) \to \mathcal{M}(\mathbb{R}), \quad F(\mu) = \mathcal{L}(UY + 1)$$

with U, Y ind. and $\mathcal{L}(Y) = \mu$.

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For $\mu, \nu \in \mathcal{M}_1(\mathbb{R})$ let

$$\ell_1(\mu, \nu) = \inf_{X,Y:\mathcal{L}(X)=\mu,\mathcal{L}(Y)=\nu} \mathbb{E}[\|X - Y\|]$$

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Show: F is a contraction according to ℓ_1 in $\mathcal{M}_1(\mathbb{R})$.

Contraction

Proof: Let X, Y s.t. $\mathcal{L}(X) = \mu, \mathcal{L}(Y) = \nu$ and

$$\mathbb{E}[|X - Y|] \leq d(\mu, \nu) + \varepsilon.$$

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Then

$$\begin{aligned} \ell_1(F(\mu), F(\nu)) &\leq \mathbb{E}[|UX + 1 - (UY + 1)|] = \mathbb{E}U\mathbb{E}[|X - Y|] \\ &\leq \mathbb{E}U(\ell_1(\mu, \nu) + \varepsilon). \end{aligned}$$

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This gives

$$\ell_1(F(\mu), F(\nu)) \leq \mathbb{E}U\ell_1(\mu, \nu)$$

□

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$$Y \stackrel{d}{=} UY + 1$$

has a unique solution in $\mathcal{M}_1(\mathbb{R})$ and it is easy to show

$$\ell_1(Y_n, Y) \to 0$$

for $\mathcal{L}(Y) = \mu$.

Process version

- Consider all $1 \leq k \leq n$ uniformly in the same algorithm $X_n(k,)$
- Rescaling gives a the r.v. $Y_n\left(\frac{k}{n}\right) = \frac{X_n(k)}{n}$ defined on lattice points $\frac{k}{n}$
- Fit Y_n to a stepfunction in $D([0,1])$
- Grübel, Rösler ('95) prove

$$Y_n \xrightarrow{d} Y$$

in $(D([0,1]), d_{Sk})$

Donsker's Theorem

Let $X_1, X_2, \ldots$ be iid r.v. with $\mathbb{E}X_1 = 0, \mathbb{E}X_1^2 = 1$ and $\mathbb{E}|X_1|^{2+\varepsilon} < \infty$ for some $\varepsilon > 0$.

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$$S^n = (S_t^n)_{t\in[0,1]}$$

with

$$S_t^n = \frac{1}{\sqrt{n}}\left(\sum_{k=1}^{\lfloor nt\rfloor} X_k + (nt - \lfloor nt\rfloor)X_{\lfloor nt\rfloor+1}\right), \quad t \in [0,1]$$

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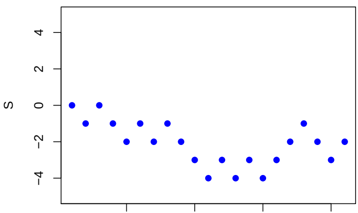

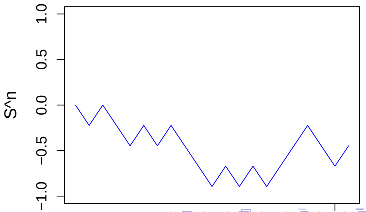

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Theorem (Donsker, 1951)

$S^n \xrightarrow{d} B$ in $(\mathcal{C}([0,1]), ||\cdot||_{\sup})$, where B is a standard Brownian Motion.

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In space:

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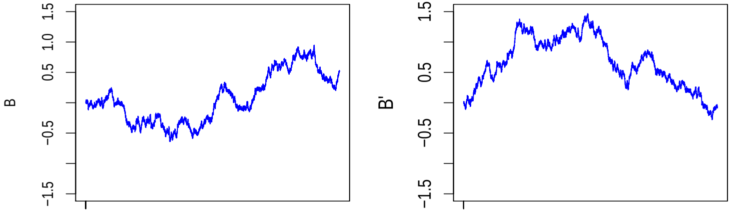

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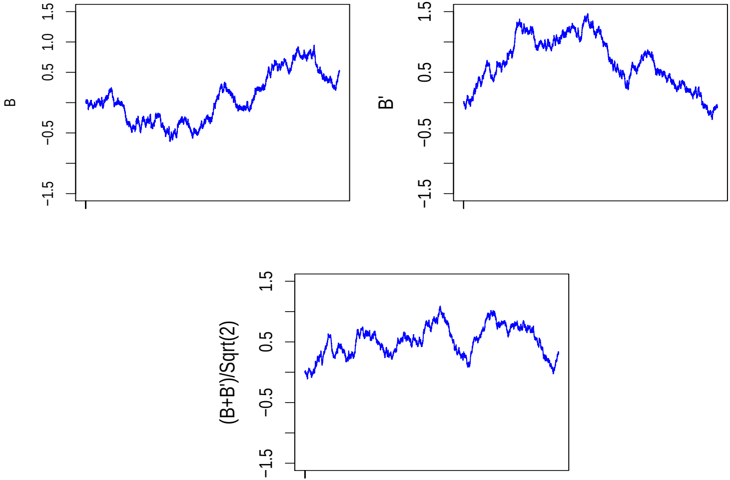

Decomposition of Brownian Motion

It holds

$$(B_t)_t \stackrel{d}{=} \left(\frac{B_t + B_t'}{\sqrt{2}} \right)_t \tag{2}$$

for two independent standard Brownian Motions B, B'.

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In general, there is no spacian decomposition of S^n.
Equation (2) is satisfied by any gaussian process.

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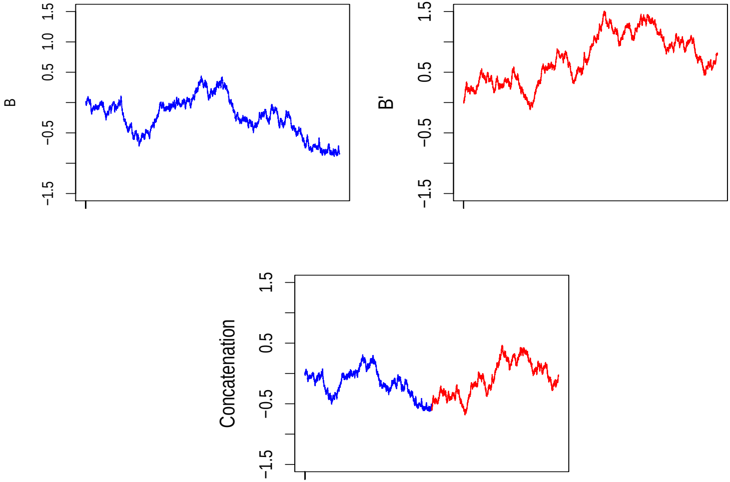

Decomposition of Brownian Motion

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\begin{aligned}
(B_t)_t &\stackrel{d}{=} \left(\frac{1}{\sqrt{2}} \left(\mathbf{1}_{\{t \le 1/2\}} B_{2t} + \mathbf{1}_{\{t > 1/2\}} B_1 \right) + \frac{1}{\sqrt{2}} \mathbf{1}_{\{t > 1/2\}} B'_{2t-1} \right)_t \\
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with (continuous and linear) functions $\varphi_\beta, \psi_\beta : \mathcal{C}([0,1]) \to \mathcal{C}([0,1])$

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\varphi_\beta(f)(t) &= \mathbf{1}_{\{t \le 1/\beta\}} f(\beta t) + \mathbf{1}_{\{t > 1/\beta\}} f(1), \\
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Theorem (Uniqueness - Characterization of Brownian Motion)

Let $X = (X_t)_{t\in[0,1]}$ be a continuous real-valued stochastic process satisfying (3). Then there exists a constant $\sigma \geq 0$ such that σX is a standard Brownian Motion.

Time-decomposition of the random walk

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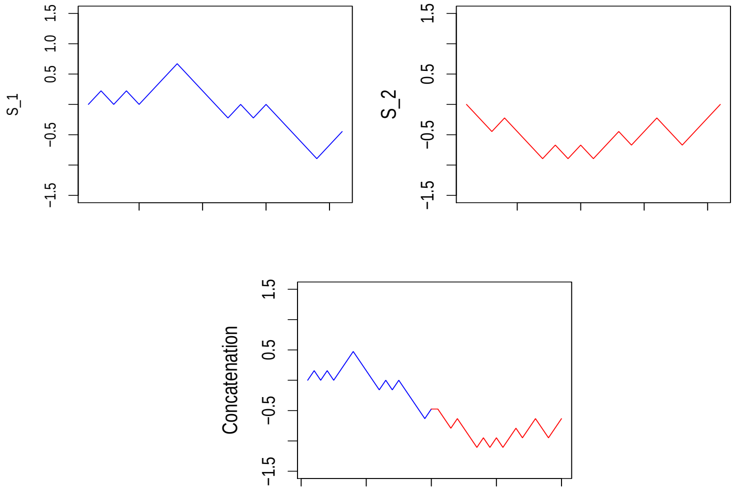

Time-decomposition of the random walk

It holds:

$$S^n \stackrel{d}{=} \sqrt{\frac{\lceil n/2 \rceil}{n}} \varphi_{\frac{n}{\lceil n/2 \rceil}} \left(S^{\lceil n/2 \rceil} \right) + \sqrt{\frac{\lfloor n/2 \rfloor}{n}} \psi_{\frac{n}{\lceil n/2 \rceil}} \left(\widehat{S}^{\lfloor n/2 \rfloor} \right), \quad (4)$$

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Now

$$\sqrt{\frac{\lceil n/2 \rceil}{n}} \to \frac{1}{\sqrt{2}}, \quad \sqrt{\frac{\lfloor n/2 \rfloor}{n}} \to \frac{1}{\sqrt{2}}.$$

suggests weak convergence $S^n \to B$.

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$$\zeta_s(\mu, \nu) = \sup_{f \in \mathcal{F}_s} |\mathbb{E}[f(X) - f(Y)]|,$$

with $\mathcal{L}(X) = \mu, \mathcal{L}(Y) = \nu$

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$$\mathcal{F}_s := \{f \in C^m(\mathcal{C}([0,1]),\mathbb{R}) : \|D^m f(x) - D^m f(y)\| \leq \|x-y\|^\alpha, \quad x, y \in \mathcal{C}$$

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Set $\zeta_s(X, Y) = \zeta_s(\mathcal{L}(X), \mathcal{L}(Y))$.

Properties of the Zolotarev distance ζ_s

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It holds $\zeta_s(X, Y) < \infty$, if

$$\mathbb{E}||X||^s, \mathbb{E}||Y||^s < \infty, \quad \mathbb{E}[g(X, \dots, X)] = \mathbb{E}[g(Y, \dots, Y)]$$

for all $k \leq m$ and multilinear, bounded functions $g : \mathcal{C}([0,1])^k \to \mathbb{R}$.
In the following assume finiteness of the considered ζ_s-distances

Properties of the Zolotarev distance ζ_s

Lemma

ζ_s is $(s,+)$ - ideal, i.e.

$$\zeta_s(\varphi(X),\varphi(Y)) \leq ||\varphi||^s \zeta_s(X,Y)$$

for any continuous and linear function $\varphi : \mathcal{C}([0,1]) \to \mathcal{C}([0,1])$ with

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Furthermore

$$\zeta_s(X_1+X_2, Y_1+Y_2) \leq \zeta_s(X_1,Y_1) + \zeta_s(X_2,Y_2)$$

for (X_1,Y_1) and (X_2,Y_2) independent.

Proof: Zolotarev ('76)

Proof of Donsker's Theorem (Sketch)

We are in the case $2 < s < 3$ and have to consider $\mathbb{E}[f(X,X)]$ for continuous, bilinear functions $f : \mathcal{C}([0,1])^2 \to \mathbb{R}$.
This is done by controlling the covariance function.

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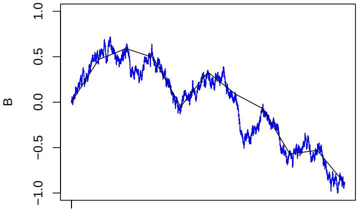
This is done by controlling the covariance function.
Since S^n and B do not share their covariance function (of course $\mathbb{E}[S^n_s, S^n_t] \to \min(s,t)$) we also consider the process B^n defined by

$$B^n_t = B_{\frac{\lfloor nt \rfloor}{n}} + (nt - \lfloor nt \rfloor)\left(B_{\frac{\lfloor nt \rfloor + 1}{n}} - B_{\frac{\lfloor nt \rfloor}{n}}\right).$$



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Donsker's Theorem follows with the following properties of Zolotarev's distance.

Properties of ζ_s in $\mathcal{C}([0,1])$

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Theorem

Let $B = \mathcal{C}([0,1])$ *and* $0 < s \leq 3$, *i.e,* $m \in \{0,1,2\}$. *Let* $(X_n)_{n\geq 1}$, X *be random variables in* $(\mathcal{C}([0,1]), \|\cdot\|)$ *where, for each* $n \geq 1$, X_n *is piecewise linear with intervals of length at least* r_n. *If*

$$\zeta_s(X_n, X) = o\left(\log^{-m}\frac{1}{r_n}\right), \quad \textit{as} \quad n \to \infty,$$

then $X_n \to X$ *in distribution.*

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$$\zeta_s(X_n, X) = o\left(\log^{-m}\frac{1}{r_n}\right), \quad \text{as} \quad n \to \infty,$$

then $X_n \to X$ in distribution.

Proof: This basically follows from a result of Barbour from the context of Stein's method.

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Theorem

Suppose $0 < s \leq 3$, *i.e,* $m \in \{0,1,2\}$. *Let* $(X_n)_{n\geq 1}$, $(Y_n)_{n\geq 1}$, Z *be random variables in* $(\mathcal{C}([0,1]), \|\cdot\|)$ *where, for each* $n \geq 1$, X_n, Y_n *are piecewise linear with intervals of length at least* r_n. *If* $Y_n \to Z$ *in distribution and*

$$\zeta_s(X_n, Y_n) = o\left(\log^{-m}\frac{1}{r_n}\right), \quad \text{as} \quad n \to \infty,$$

then $X_n \to Z$ *in distribution.*