Optimization of polling systems with batch service

Jan-Pieter Dorsman, Rob van der Mei, Erik Winands

Technische Universiteit Eindhoven University of Technology

CWI Centrum Wiskunde & Informatica

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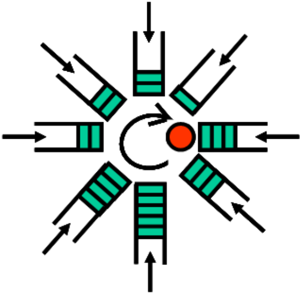
The polling model

A queueing system consisting of a number of queues and a single server, where there is a switch-over time when the server moves between queues.



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- Manufacturing systems
- Computer communication systems
- Traffic systems

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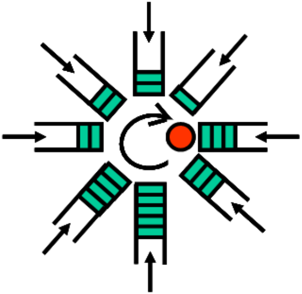

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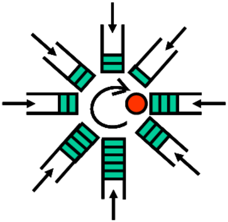
1. The order in which to serve the queues;
2. The number of customers to be served during a service period;
3. The order in which customers within each queue are served.



We will limit the analysis to the most common configuration:

1. Cyclic service
2. Exhaustive (server will switch iff queue is empty)
3. First come first served

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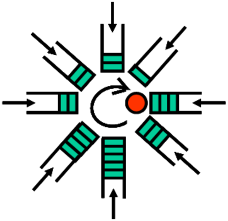

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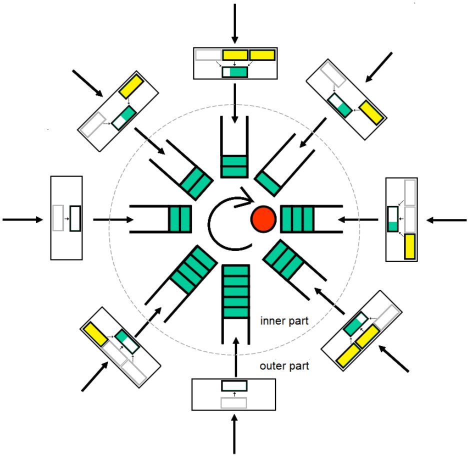
Extended model

We consider an extended version of this model.

Assumption

Processing time of a type-i batch is independent of its size!



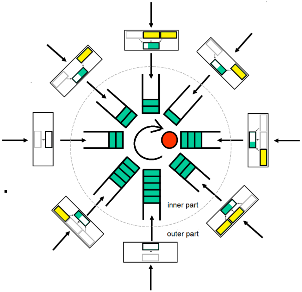
Examples:

- Oven with a fixed number of baking slots,
- Paint bath which paints several items at once.

A trade-off arises...

The waiting time of a customer can be decomposed in two parts:

- V_i, the time a type-i customer waits until its type-i batch is fully accumulated,
- W_i, the time a type-i customers waits in the inner system as part of a type-i batch.



We are confronted with a trade-off when choosing batch sizes. D_i is the size of type-i batches:

- When taking D_i small, V_i will be low, but W_i will be high!
- When taking D_i large, W_i will be low, but V_i will be high!

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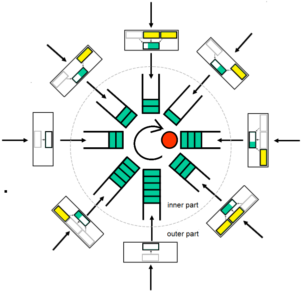

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Problem description

The problem at hand

How to take batch sizes $\vec{D} = (D_1, \ldots, D_N)$ such that the weighted sum of the mean *total* waiting times is minimized?

We need to define a cost function:

$$C(\vec{D}) = \sum_{i=1}^{N} c_i(\mathbb{E}[V_i] + \mathbb{E}[W_i]).$$

Let E_{ij} be the event that an arriving type-i customer is the j^{th} taking place in the currently accumulating type-i batch.

$\mathbb{E}[V_i]$ is then easily computed by using

$$\mathbb{P}[E_{ij}] = \frac{1}{D_i} \text{ and } \mathbb{E}[V_i|E_{ij}] = \frac{D_i - j}{\lambda_i},$$

We end up with

$$C(\vec{D}) = \sum_{i=1}^{N} c_i \left(\frac{D_i - 1}{2\lambda_i} + \mathbb{E}[W_i] \right).$$

Problem description

The problem at hand

$$\arg\min_{\vec{D}} \sum_{i=1}^{N} c_i \left(\frac{D_i - 1}{2\lambda_i} + \mathbb{E}[W_i] \right)$$

No exact expression or even numerical method available to compute $\mathbb{E}[W_i]$. Two approaches possible:

- Approach 1: Use approximation by Boon et al., and approximate solution numerically,
- Approach 2: Use an even simpler approximation, $\mathbb{E}[W_{i,app}]$, and gain a closed-form approximation.

Boon et al.

Boon, Winands, Adan and Van Wijk (2009) provide an approximation of the mean waiting time for general load:

$$\mathbb{E}[W_{i,Boon}] = \frac{K_0 + K_{1,i}\rho + K_{2,i}\rho^2}{1-\rho},$$

where K_0, $K_{1,i}$ and $K_{2,i}$ are closed-form functions dependent on the polling system parameters and determined by three requirements:

Light traffic requirements:

1. $\mathbb{E}[W_{i,Boon}]|_{\rho=0} = \mathbb{E}[W_i]|_{\rho=0}$
2. $\frac{d}{d\rho}\mathbb{E}[W_{i,Boon}]|_{\rho=0} = \frac{d}{d\rho}\mathbb{E}[W_i]|_{\rho=0}$

Heavy traffic requirement:

3. $(1-\rho)\mathbb{E}[W_{i,Boon}]|_{\rho=1} = (1-\rho)\mathbb{E}[W_i]|_{\rho=1}$

Numerical Approach

Solve

$$\arg\min_{\vec{D}} C_{Boon}(\vec{D}) = \arg\min_{\vec{D}} \sum_{i=1}^{N} c_i \left(\frac{D_i - 1}{2\lambda_i} + \mathbb{E}[W_{i,Boon}] \right)$$

numerically and round the obtained values.

Pro's:

- Very good performance.

Con's:

- Computation time may grow infeasibly long when the number of queues/dimensions increases.
- The numerical approach only gives limited insight in how the optimal batch sizes behave in the system's parameters.
- Implementation may be quite cumbersome.

Closed-form approach

$\mathbb{E}[W_{i,Boon}]$ is too complex to gain a nice closed-form approximation of optimal batch sizes.
Therefore, we use a simpler approximation of $\mathbb{E}[W_i]$ with the form:

$$\mathbb{E}[W_{i,app}] = \frac{a + b_i \rho}{1 - \rho}$$

Coefficients a and b_i are determined by two requirements:

- LT: $\mathbb{E}[W_{i,app}]|_{\rho=0} = \mathbb{E}[W_i]|_{\rho=0}$
- HT: $(1-\rho)\mathbb{E}[W_{i,app}]|_{\rho=1} = (1-\rho)\mathbb{E}[W_{i,simplification}]|_{\rho=1}$

We end up with the following approximative cost function

$$C_{app}(\vec{D}) = \sum_{i=1}^{N} c_i \left(\frac{D_i - 1}{2\lambda_i} + \mathbb{E}[W_{i,app}] \right)$$

Closed-form approach

Solve

$$\arg\min_{\vec{D}} C_{app}(\vec{D}) = \arg\min_{\vec{D}} \sum_{i=1}^{N} c_i \left(\frac{D_i - 1}{2\lambda_i} + \mathbb{E}[W_{i,app}] \right)$$

with ordinary calculus.

- The one-dimensional case is easily solved.
- For the multi-dimensional case, the system of equations $\frac{d}{dD_i} C_{app}(\vec{D}) = 0 \ \forall i$ does not result in nice expressions.

Work-around: reduce multi-dimensional problem to a one-dimensional problem by a priori assuming ratios between optimal batch sizes.

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As batch sizes increase, the load of the inner system drops rapidly, hence $\mathbb{E}[W_{i,app}]$ will be near-insensitive to the batch sizes.

The *ratios* of the optimal variables of the following problem *can* be determined:

$$\arg\inf_{\vec{D}} C_{app}(\vec{D}) = \arg\inf_{\vec{D}} \sum_{i=1}^{N} c_i \frac{D_i - 1}{2\lambda_i}.$$

I.e., the numbers $d_2 = \frac{D_2^{opt}}{D_1^{opt}}, \ldots, d_N = \frac{D_N^{opt}}{D_1^{opt}}$ are known here:

$$d_i = \frac{\lambda_i}{\lambda_1}\sqrt{\frac{c_1\mathbb{E}[B_i]}{c_i\mathbb{E}[B_1]}}$$

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Closed-form approach

Back to our problem, we write $D_1^{opt} = E, D_2^{opt} = d_2 E, \ldots, D_N^{opt} = d_N E$ with the d_i as found in the simple problem. Only optimization in E is needed:

$$E_{app}^{opt} = \sum_{i=1}^{N} \frac{\lambda_i \mathbb{E}[B_i]}{d_i} + \sqrt{2 \left(\sum_{i=1}^{N} \frac{c_i d_i}{\lambda_i} \right)^{-1} \left(\sum_{i=1}^{N} c_i \omega_{i,app} \right) \left(\sum_{i=1}^{N} \frac{\lambda_i \mathbb{E}[B_i]}{d_i} \right)}.$$

Going back to optimal batch sizes by multiplying with the d_i, we have:

$$\begin{aligned}\vec{D}_{app}^{opt} =& (D_{1,app}^{opt}, D_{2,app}^{opt}, \ldots, D_{N,app}^{opt}) \\ =& (E_{app}^{opt}, \frac{\lambda_2}{\lambda_1} \sqrt{\frac{c_1 \mathbb{E}[B_2]}{c_2 \mathbb{E}[B_1]}} E_{app}^{opt}, \ \ldots \ , \frac{\lambda_N}{\lambda_1} \sqrt{\frac{c_1 \mathbb{E}[B_N]}{c_N \mathbb{E}[B_1]}} E_{app}^{opt}).\end{aligned}$$

Proper rounding of these values gives a nicely defined and well-performing closed-form approximation.

Closed-form approach

The obtained closed-form approximation...

Pro's:

- Performs increasingly well when the number of queues or switch-over times get larger,
- Is easily implemented and requires virtually no computation time,
- Gives insights in the dependency of the optimal batch sizes on the systems parameters.

Con's:

- Performs not as well as the numerical approach in case of small numbers of queues or small switch-over times. Complementary effect!

Validation

Validation by means of simulation, based on testbeds of symmetric and asymmetric systems, with a total of 1260 systems.

Differences expressed relatively in terms of costs in %.

Overall:

Test bed	Numerical	Closed-form
Symmetric	0.303	1.624
Asymmetric	0.645	4.848

Categorized in the number of queues:

Test bed	N=2		N=5	
	Numerical	Closed-form	Numerical	Closed-form
Symmetric	0.428	2.776	0.178	0.471
Asymmetric	0.870	7.715	0.419	1.981

Further Research

The current analysis may be extended in multiple directions:

- Optimization based on higher moments and tail probabilities,
- Model variations: non-cyclic routing, other service disciplines, size-dependent service requirements, introduction of the possibility to idle, etc.
- Rounding strategy of the fractional D_i-values.