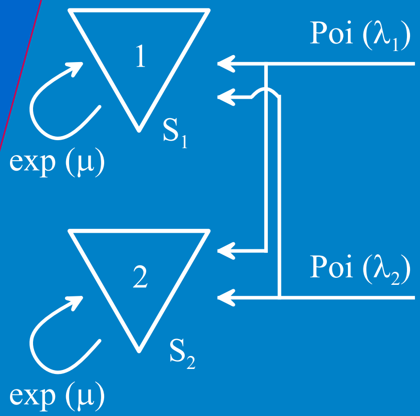
Optimal lateral transshipment policies in spare parts inventory models



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Technische Universiteit **Eindhoven** University of Technology

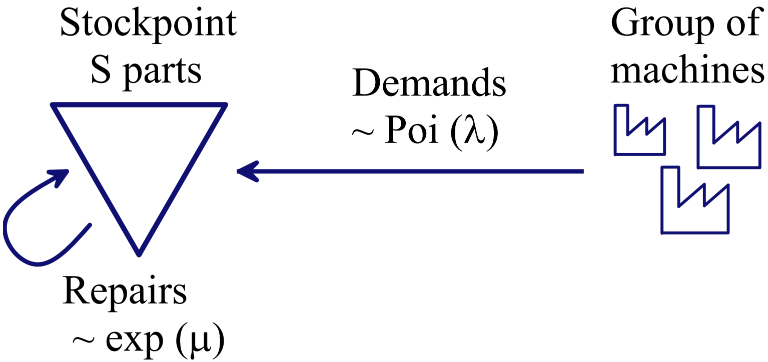
YEQT-IV: Optimal Control in Stochastic Systems

EURANDOM, Eindhoven, November 27, 2010

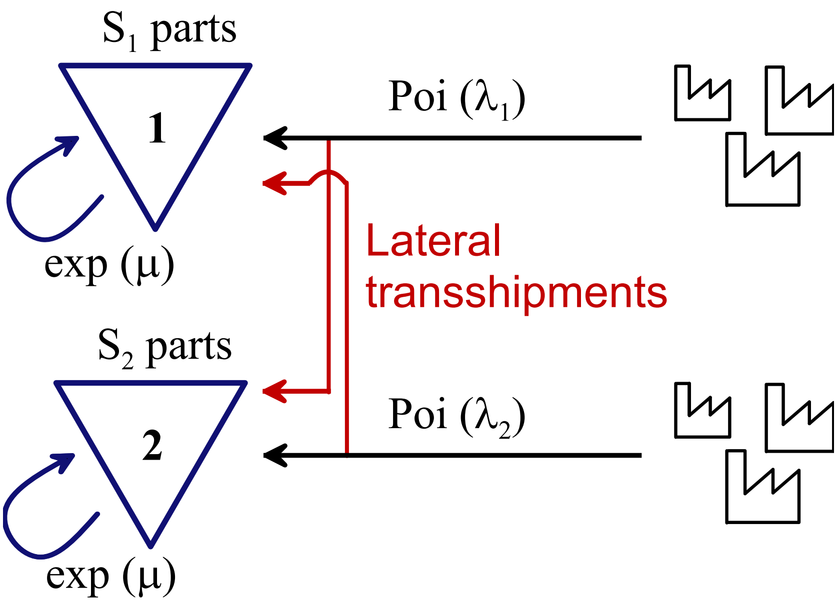
Introduction

Spare Parts Inventory System

- Technically advanced machines: down-times extremely expensive
- Breakdown: demand for spare part
- Ready-for-use spare parts are kept on stock: repair-by-replacement strategy
- Broken parts returned and repaired
- No back-orders: emergency repair procedure



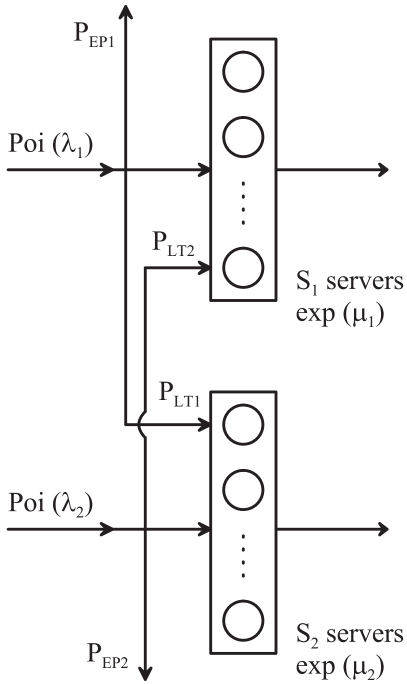
Two stock points: pooling of inventory

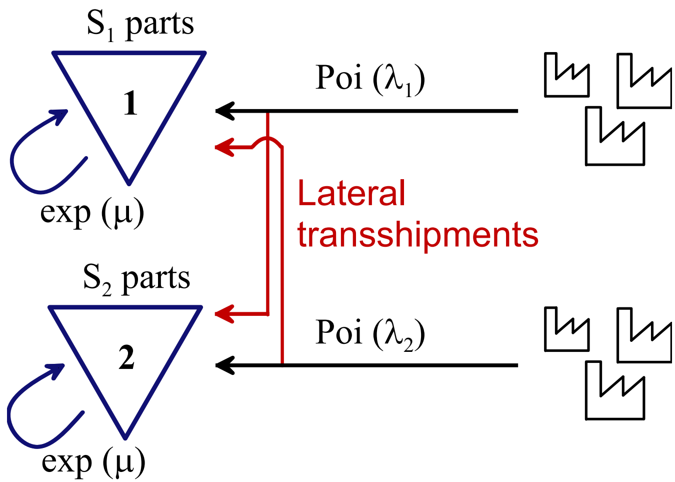


Lateral transshipment costs are small (compared to emergency procedure costs), hence costs can be saved.

Introduction

PhD project: "Creation of Pooling in Queueing and Inventory Systems"





Two location lateral transshipment problem

How to route the demands?

- Directly from stock;
- Via lateral transshipment (penalty costs P_{LT_i});
- Via emergency procedure (penalty costs P_{EP_i}).

Structure optimal lateral transshipment policy?

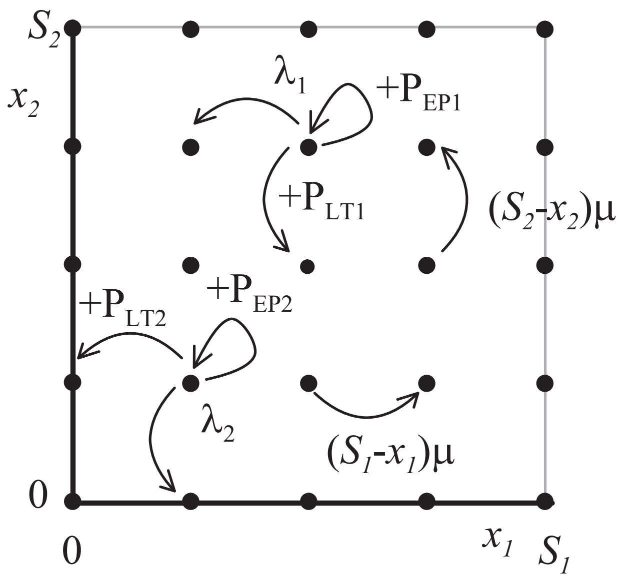
When are simple policies optimal?

Markov Decision Problem

MDP with states (stock levels): (x_1, x_2)

Events:

- demand at 1 (decision!)
- demand at 2 (decision!)
- replenishment at 1
- replenishment at 2



Technique

Value iteration

Value function V_n (minimal expected n-period costs function, $V_0 \equiv 0$):

$$V_{n+1}(x_1, x_2) = \frac{\lambda_1 H_1(V_n) + \lambda_2 H_2(V_n) + \mu_1 G_1(V_n) + \mu_2 G_2(V_n)}{\lambda_1 + \lambda_2 + S_1\mu_1 + S_2\mu_2}.$$

Operators

H_i demands at i, G_i replenishments at i

$$H_1 f(x_1, x_2) = \begin{cases} \min\{f(x_1 - 1, x_2), \\ \qquad f(x_1, x_2 - 1) + P_{LT_1}, \\ \qquad f(x_1, x_2) + P_{EP_1}\}, & \text{if } x_1 > 0, x_2 > 0. \\ \ldots \end{cases}$$

$$G_1 f(x_1, x_2) = (S_1 - x_1) f(x_1 + 1, x_2) + x_1 f(x_1, x_2).$$

Technique

Theorem

Provided equal μ's, V_n is multimodular for all $n \geq 0$.

Multimodularity (for 2 dimensions):

Supermodularity: $f(x_1, x_2) + f(x_1 + 1, x_2 + 1) \geq f(x_1 + 1, x_2) + f(x_1, x_2 + 1)$,
Superconvexity(1,2): $f(x_1 + 2, x_2) + f(x_1, x_2 + 1) \geq f(x_1 + 1, x_2) + f(x_1 + 1, x_2 + 1)$,
Superconvexity(2,1): $f(x_1, x_2 + 2) + f(x_1 + 1, x_2) \geq f(x_1, x_2 + 1) + f(x_1 + 1, x_2 + 1)$,
which implies:
Convexity(1): $f(x_1, x_2) + f(x_1 + 2, x_2) \geq 2f(x_1 + 1, x_2)$,
Convexity(2): $f(x_1, x_2) + f(x_1, x_2 + 2) \geq 2f(x_1, x_2 + 1)$.

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Theorem

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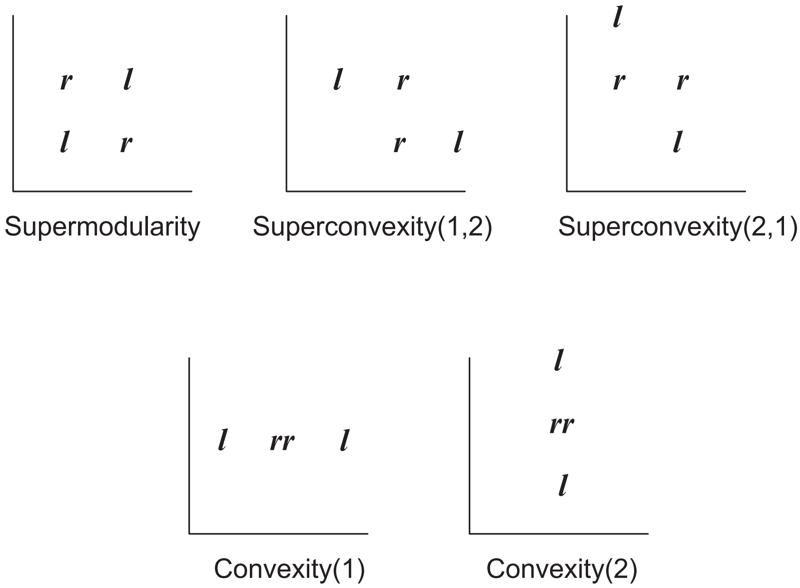
Multimodularity (for 2 dimensions): $l \geqslant r$



Theorem

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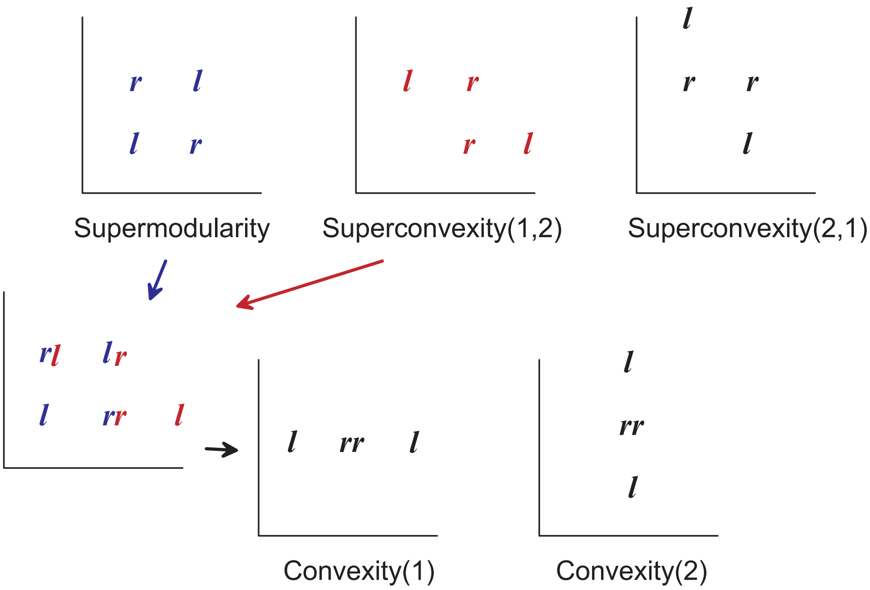
Multimodularity (for 2 dimensions): $l \geqslant r$



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Technique

Theorem

Provided equal μ's, V_n is multimodular for all $n \geq 0$.

Proof

By induction:

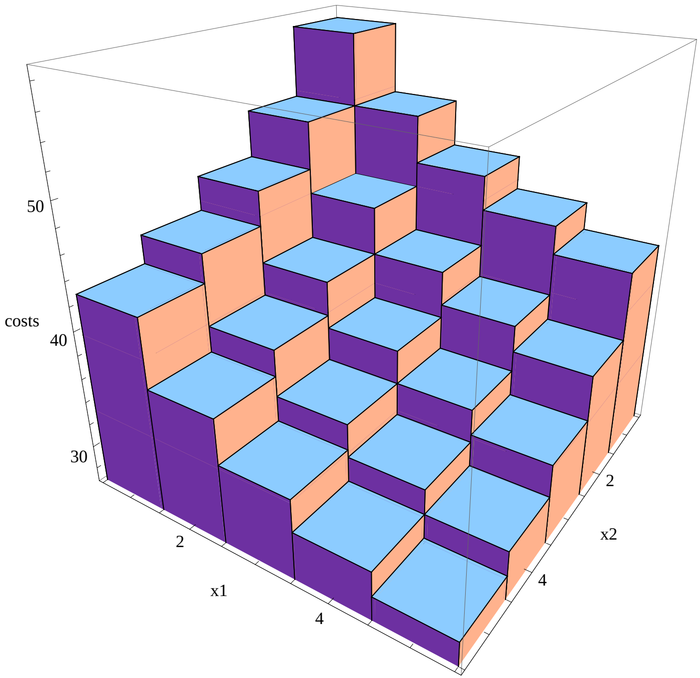
- $V_0 \equiv 0$ is MM
- Prove that H_1, H_2, and $G_1 + G_2$ preserve multimodularity:
  - f MM $\Rightarrow H_1 f$ MM
  - f MM $\Rightarrow H_2 f$ MM
  - f MM $\Rightarrow (G_1 + G_2) f$ MM

Hence V_n is multimodular for all $n \geq 0$.

Example Value function

Symmetric parameters: $S = 4$, $\lambda = 1$, $\mu = 1/3$, $P_{EP} = 10$, $P_{LT} = 7$.

V_{50}:



Implications of structural properties V_n

Consider e.g. $(x_1, 0)$, with $x_1 > 0$.

Decision for demand at 1: EP or DI.

$P_{EP_1} + V_n(x_1, 0) - V_n(x_1 - 1, 0)$ is *increasing* in x_1, as V_n is Convex in x_1.

So there exists a threshold, say $T^{DI}(0)$, such that

- for $x_1 < T^{DI}(0)$: emergency procedure (EP) is optimal,
- for $x_1 \geq T^{DI}(0)$: directly from stock (DI) is optimal.

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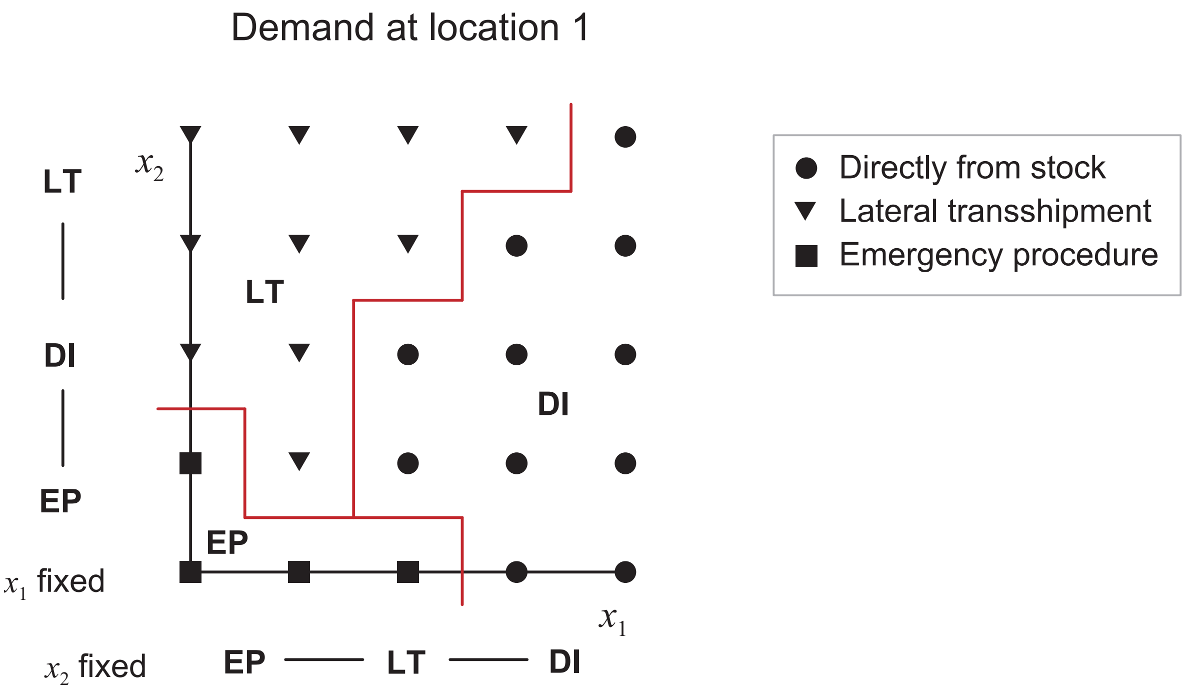
Structure of Optimal Policy

The optimal policy is a threshold type policy.



Simple policies

- Complete pooling

  (always hand out parts in case of demands, LTs)

Optimal if: $$\begin{cases} P_{LT_1} + \frac{\lambda_2}{\lambda_2+\mu} P_{EP_2} \leq P_{EP_1}, \\ P_{LT_2} + \frac{\lambda_1}{\lambda_1+\mu} P_{EP_1} \leq P_{EP_2}. \end{cases}$$

- Hold back policy

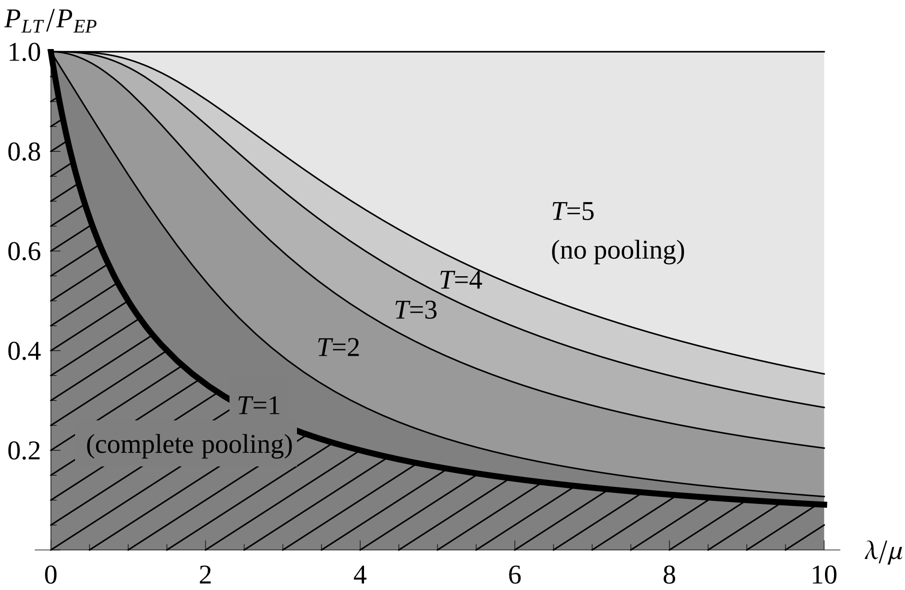
  (always hand out parts in case of demands,

  hold back parts in case of LTs: hold back levels T_1, T_2)

Optimal if: $$\begin{cases} P_{EP_2} \leq P_{LT_2} + \left(1 + \frac{\mu}{\lambda_2}\right) P_{EP_1}, \\ P_{EP_1} \leq P_{LT_1} + \left(1 + \frac{\mu}{\lambda_1}\right) P_{EP_2}. \end{cases}$$

Symmetric Parameters

- Hold back policy is optimal
- Complete pooling optimal if $\frac{P_{LT}}{P_{EP}} \leq \frac{\mu}{\lambda/\mu}$.



Lateral Transshipment Problem

Model extensions:

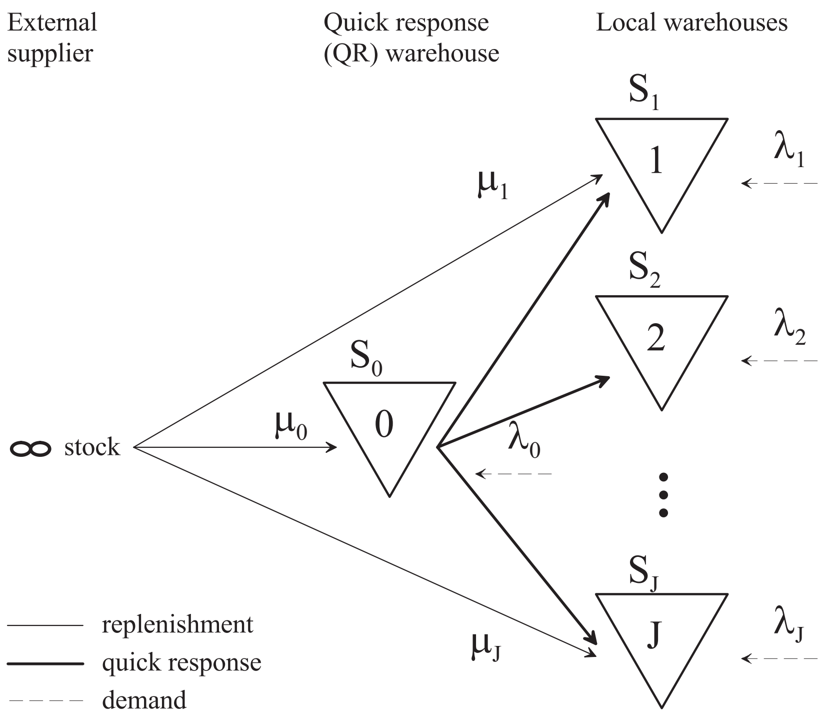
- Consumables (holding costs)
- LTs in one direction
- Asymmetric repair rates
- Limited repair capacity

$\rightarrow$ More than two stock points ??

- Approximation algorithm (hold back policy)
- Related model

Inventory model with a quick response warehouse

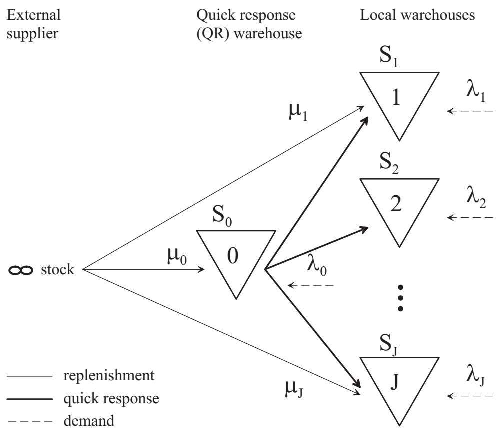
related problem, same techniques



Inventory model with a quick response warehouse

Decisions:

- Stock-out at local warehouse j:
  - quick response P_j^{QR}, or
  - emergency procedure P_j^{EP}?
- Demand at QR warehouse:
  - satisfy, or
  - emergency procedure P_0^{EP}?



Inventory model with a quick response warehouse

Simple policies optimal:

- Always quick response when j is stocked-out. Optimal if:

$$\lambda_0 P_0^{EP} + \sum_{k=1}^{J} \lambda_k (P_k^{EP} - P_k^{QR}) \leq (P_j^{EP} - P_j^{QR}) \Big(\mu_0 + \sum_{k=0}^{J} \lambda_k \Big)$$

- Always satisfy demand at QR warehouse. Optimal if:

$$\sum_{k=1}^{J} \lambda_k (P_k^{EP} - P_k^{QR}) \leq P_0^{EP} \Big(\mu_0 + \sum_{k=1}^{J} \lambda_k \Big)$$

Model

- Markov Decision Problem (MDP)

  state (stock levels): $x = (x_0, x_1, \ldots, x_J)$

- Event Based Dynamic Programming

  value function V_n:

$$V_{n+1}(x) = \frac{1}{\sum_{j=0}^{J} S_j \mu_j + \sum_{j=0}^{J} \lambda_j} \Big(\sum_{j=0}^{J} \mu_j G_j V_n(x) + \sum_{i=1}^{J} \lambda_j H_j V_n(x) + \lambda_0 H_{QR} V_n(x) \Big)$$

  - G_j replenishment at j $(0..N)$
  - H_j demand at j $(1..N)$
  - H_{QR} demand at QR warehouse

Value function

$$V_{n+1}(x) = \frac{1}{\sum_{j=0}^{J} S_j \mu_j + \sum_{j=0}^{J} \lambda_j} \Big(\sum_{j=0}^{J} \mu_j G_j V_n(x) + \sum_{i=1}^{J} \lambda_j H_j V_n(x) + \lambda_0 H_{QR} V_n(x) \Big)$$

- $G_j f(x) = \begin{cases} (S_j - x_j) f(x + e_j) + x_j f(x) & \text{if } x_j < S_j; \\ S_j f(x) & \text{if } x_j = S_j. \end{cases}$
- $H_j f(x) = \begin{cases} f(x - e_j) & \text{if } x_j > 0; \\ \min\{P_j^{QR} + f(x - e_0), \\ \qquad P_j^{EP} + f(x)\} & \text{if } x_j = 0, x_0 > 0; \\ P_j^{EP} + f(x) & \text{otherwise.} \end{cases}$
- $H_{QR} f(x) = \begin{cases} \min\{f(x - e_0), P_0^{EP} + f(x)\} & \text{if } x_0 > 0; \\ P_0^{EP} + f(x) & \text{if } x_0 = 0. \end{cases}$

Structural properties

Convexity and Supermodularity:

$$\begin{aligned} \text{Conv}(x_i): &\quad f(x) + f(x + 2e_i) \geq 2f(x + e_i), \\ \text{Supermod}(x_i, x_j): &\quad f(x) + f(x + e_i + e_j) \geq f(x + e_i) + f(x + e_j) \text{ for } i \neq j. \end{aligned}$$

Theorem: V_n is Conv and Supermod for all $n \geq 0$ when V_0 is so.

Proof: By induction, as Conv and Supermod are preserved by G_j, H_j, and H_{QR}.

Consequently: Optimal policy is threshold type policy, and we can derive conditions under which simple policies optimal.

Extensions

Analogous results for:

- including holding costs
- backlogging at local warehouses
- state-dependent replenishment rates

Further work

numerical study: optimal policy vs. simple policies

Summary

- Two location lateral transshipment model
- Quick response warehouse model

$\rightarrow$ optimal policy structure
& conditions for simple policies

A.C.C. van Wijk, I.J.B.F. Adan and G.-J. van Houtum,

- Optimal Lateral Transshipment Policy for a Two Location Inventory Problem (Eurandom report # 2009-027).
- Approximate Evaluation of Multi-Location Inventory Models with Lateral Transshipments and Hold Back Levels (in preperation).
- Optimal Policy for a Multi-location Inventory System with a Quick Response Warehouse (in preperation).