Diffusive bounds and central limit theorem for the true (myopic) self-avoiding walk in $d \geq 3$

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Outline

joint work with

- Bálint Tóth (professor, Budapest, PhD advisor)
- Bálint Vető (PhD student, Budapest)

Myopic (or 'true') self-avoiding walk (TSAW)

D. Amit, G. Parisi, L. Peliti, 1983

$X(t)$ continuous time nearest neighbour random walk on $\mathbb{Z}^d$

Local time (occupation time measure) with initialization:
$l(t,x) := l(0,x) + |\{s \in [0,t] : X(s) = x\}|$

Jump rates:

$$\begin{aligned}\mathbf{P}\left(X(t+\mathrm{d}t) = y \,\middle|\, \mathrm{past}, X(t) = x\right) \\ = \mathbb{1}_{\{|y-x|=1\}} w(l(t,x) - l(t,y))\,\mathrm{d}t + o(\mathrm{d}t)\end{aligned}$$

where $w : \mathbb{R} \to [0,\infty)$ increasing.

The walker is pushed by the discrete negative gradient of its own local time to less visited areas.

Similar models, earlier results and conjectures

Variations: edge/site repulsion, discrete/continuous time.

Dimension-dependent behaviour:

$d = 1$ $X(t) \sim t^{2/3}$ with difficult non-Gaussian scaling limit

- limit theorem for an edge-repulsion version of TSAW (B. Tóth, 1995)
- construction of the limit process (B. Tóth, W. Werner, 1998)
- a site-repulsion version of TSAW (B. Tóth, B. Vető, 2009)

$d = 2$ $X(t) \sim t^{1/2}(\log t)^{\xi}$ with Gaussian limit, $\xi = ?$

- partial results (B. Valkó, 2009)

$d \geq 3$ $X(t) \sim t^{1/2}$ with Gaussian limit

- limit theorem for a site-repulsion version of TSAW (I. H., B. Tóth, B. Vető, 2010)

Self-repelling Brownian polymer model

Continuous space variant: Self-repelling Brownian polymer model
J. Norris, C. Rogers, D. Williams, 1987
R. Durrett, C. Rogers, 1992

$X(t)$ diffusion process in $\mathbb{R}^d$

occupation time measure with initialization:
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$V : \mathbb{R}^d \to \mathbb{R}^+$ approximate identity, e.g. $V(x) = e^{-|x|^2}$
Evolution (smeared-out local times):

$$\mathrm{d}X(t) = \mathrm{d}B(t) - \mathrm{grad}(V * l(t, \cdot))(X(t))\,\mathrm{d}t$$

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CLT for the SRBP in $d \geq 3$ (I. H., B. Tóth, B. Vető, 2009)

Environment seen from the position of the walker

$X(t)$ random walk on $\mathbb{Z}^d$ with jump rates

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where $w:\mathbb{R}\to[0,\infty)$ and the local time is

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The environment seen by the walker:

$$\eta(t) = \big(\eta(t,x)\big)_{x\in\mathbb{Z}^d}, \qquad \eta(t,x) := l(t, X(t)+x).$$

$t\mapsto\eta(t)$ is a Markov process on the state space

$$\Omega := \{\omega = \big(\omega(x)\big)_{x\in\mathbb{Z}^d} \;:\; \omega(x)\in\mathbb{R}\}$$

Assumptions

(1) $d \geq 3$.

(2) $\inf_{u\in\mathbb{R}} w(u) = \gamma > 0$.

(3) $\inf_{u\in\mathbb{R}} r'(u) = c_1 > 0$.
s and r denote the even, respectively, odd part of w:

$$s(u) := \frac{w(u) + w(-u)}{2} - \gamma, \qquad r(u) := \frac{w(u) - w(-u)}{2},$$

(4) w is a polynomial.

Infinitesimal generator, stationary measure

The infinitesimal generator of the process $t \mapsto \eta(t)$ is acting as an operator on smooth cylinder functions from $\mathcal{L}^2(\Omega, \pi)$:

$$Gf(\omega) = \partial_0 f(\omega) + \sum_{e \in \mathbb{Z}^d, |e|=1} w(\omega(0) - \omega(e))\big(f(\tau_e \omega) - f(\omega)\big),$$

where

$$\partial_0(\omega) := \frac{\partial f}{\partial \omega(0)}(\omega),$$

and

$$\tau_z : \Omega \to \Omega, \qquad \tau_z \omega(x) := \omega(z + x),$$

is the group of spatial shifts, acting naturally on Ω.

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The stationary measure is

$$d\pi(\omega) = Z^{-1} \exp\Big\{ -\frac{1}{2} \sum_{\substack{x,y \in \mathbb{Z}^d \\ |x-y|=1}} R(\omega(x) - \omega(y)) \Big\} \mathrm{d}\omega,$$

where $R(u) := \int_0^u r(v)\mathrm{d}v$.

Main results

Theorem

$\eta(t)$ is a stationary and ergodic Markov process on (Ω, π).

Corollary

For π-almost all initial profiles

$$X(t)/t \to 0.$$

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Theorem (I. H., B. Tóth, B. Vető, 2010)

Under the previous assumptions,

$$0 < \gamma \leq \inf_{\substack{e \in \mathbb{R}^d \\ |e|=1}} \varliminf_{t\to\infty} t^{-1}\mathbf{E}\left((e \cdot X(t))^2\right) \leq$$
$$\leq \sup_{\substack{e \in \mathbb{R}^d \\ |e|=1}} \varlimsup_{t\to\infty} t^{-1}\mathbf{E}\left((e \cdot X(t))^2\right) < \infty.$$

Main results

Theorem (I. H., B. Tóth, B. Vető, 2010)

If we also assume that $r(u)$ is linear, the degree of $s(u)$ is at most 4 and the main coefficient small enough, then the matrix of asymptotic covariances

$$\sigma^2 = (\sigma^2_{kl})_{1\le k,l\le d}, \quad \sigma^2_{kl} := \lim_{t\to\infty} t^{-1}\mathbf{E}\left(X_k(t)X_l(t)\right)$$

exists and it is nondegenerate. The finite dimensional distributions of the rescaled displacement process

$$X_N(t) := N^{-1/2}X(Nt),$$

converge to those of a d-dimensional Brownian motion with covariance matrix σ^2.

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Remark

If $r(u)$ is linear, the stationary measure is Gaussian.

Self-adjoint and skew-self-adjoint parts of the generator

Denote

$$S := -\frac{1}{2}(G + G^*), \qquad A := \frac{1}{2}(G - G^*),$$

the self-adjoint, respectively, skew-self-adjoint parts of the infinitesimal generator.

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$$Sf(\omega) = -\gamma\Delta - \sum_{\substack{e \in \mathbb{Z}^d \\ |e|=1}} s(\omega(0) - \omega(e))\big(T_e f(\omega) - f(\omega)\big)$$

$$Af(\omega) = \partial_0 f(\omega) + \sum_{\substack{e \in \mathbb{Z}^d \\ |e|=1}} r(\omega(0) - \omega(e))\big(T_e f(\omega) - f(\omega)\big),$$

where

$$T_z f(\omega) := f(\tau_z \omega), \qquad \Delta := \sum_{\substack{e \in \mathbb{Z}^d \\ |e|=1}} (T_e - I).$$

Yaglom-reversibility, ergodicity

Stationarity follows from the so-called *Yaglom-reversibility*:

$$JSJ = S, \qquad JAJ = -A, \qquad JGJ = G^*$$

where J is the unitary involution

$$Jf(\omega) := f(-\omega)$$

on $\mathcal{H} := \mathcal{L}^2(\Omega, \pi)$.

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Ergodicity follows from

$$S \geq -\gamma\Delta$$

and ergodicity of the shifts on (Ω, π).

Diffusive lower bound

Consider the following martingale+compensator decomposition:

$$X(t) = N(t) + M(t) + \int_0^t \varphi(\eta(s))\mathrm{d}s,$$

where $N(t)$ is the martingale part due to the jump rates γ, $M(t)$ is the martingale part due to the jump rates $(w - \gamma)$ and $\varphi : \Omega \mapsto \mathbb{R}^d$ is the compensator (conditional velocity) function:

$$\varphi_l(\omega) = w(\omega(0) - \omega(e_l)) - w(\omega(0) - \omega(-e_l))$$

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$$\varphi_l(\omega) = w(\omega(0) - \omega(e_l)) - w(\omega(0) - \omega(-e_l))$$

Direct calculations show that $N(t)$ is uncorrelated with the other terms.

Diffusive upper bound

The martingale terms scale diffusively, so the task is to prove diffusive upper bound for the integral term:

$$\varlimsup_{t\to\infty} t^{-1}\mathbf{E}\left(\Big(\int_0^t \varphi(\eta(s))\mathrm{d}s\Big)^2\right) < \infty.$$

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From standard variational arguments (e.g. Sethuraman – Varadhan – Yau, 2000),

$$\varlimsup_{t\to\infty} t^{-1}\mathbf{E}\left(\Big(\int_0^t \varphi(\eta(s))\mathrm{d}s\Big)^2\right) \le$$
$$\le \varlimsup_{t\to\infty} t^{-1}\mathbf{E}\left(\Big(\int_0^t \varphi(\xi(s))\mathrm{d}s\Big)^2\right),$$

where $\xi(s)$ is the (reversible) process with generator S and the same stationary distribution.

Fourier-transform

The usual way to write this is

$$\int_{\Omega} \varphi(\omega)(S^{-1}\varphi)(\omega)\mathrm{d}\pi(\omega) < \infty.$$

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Denote

$$C(x) := \mathbf{E}\left(\varphi(\omega)\varphi(\tau_x\omega)\right),$$
$$\hat{C}(p) := \sum_{x\in\mathbb{Z}^d} e^{ip\cdot x}C(x), \quad p \in [-\pi,\pi]^d.$$

In $d \geq 3$,

$$\sup_{p\in[-\pi,\pi]^d} \hat{C}(p) < \infty.$$

is sufficient for the upper bound to hold.

Brascamp–Lieb-inequality

The following important consequence of the Brascamp–Lieb-inequality is used:

Lemma

Let $f:\mathbb{R}\to\mathbb{R}$ be smooth, $e\in\mathbb{Z}^d, |e|=1$ fixed and denote

$$
\begin{aligned}
C(x) &:= \mathbf{Cov}\left(f(\omega(0)-\omega(e)), f(\omega(x)-\omega(x+e))\right), \\
C'(x) &:= \mathbf{Cov}\left(f'(\omega(0)-\omega(e)), f'(\omega(x)-\omega(x+e))\right). \\
m' &:= \mathbf{E}\left(f'(\omega(0)-\omega(e))\right).
\end{aligned}
$$

Then

$$
\sup_{p\in[-\pi,\pi]^d} \hat{C}(p) \le (m')^2 + d^{-1} \sup_{p\in[-\pi,\pi]^d} \hat{C}'(p).
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The following important consequence of the Brascamp–Lieb-inequality is used:

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Let $f : \mathbb{R} \to \mathbb{R}$ *be smooth,* $e \in \mathbb{Z}^d, |e| = 1$ *fixed and denote*

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The proof for the diffusive upper bound is finished by induction on the degree of $w(x)$.

Kipnis – Varadhan theory

In the decomposition

$$X(t) = N(t) + M(t) + \int_0^t \varphi(\eta(s))\mathrm{d}s,$$

$N(t)$ and $M(t)$ are stationary, ergodic martingales and the martingale CLT can be applied. CLT for the integral term has been originally investigated by Kipnis and Varadhan.

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In general form: $\eta(t)$ is a stationary and ergodic Markov process on the state space (Ω, π). G is the infinitesimal generator of $\eta(t)$ acting on $\mathcal{L}^2(\Omega, \pi)$. $\varphi \in \mathcal{L}^2(\Omega, \pi)$ with $\int_\Omega \varphi \, \mathrm{d}\pi = 0$.

Question

Sufficient condition for the martingale approximation and central limit theorem for

$$Y_N(t) := \frac{1}{\sqrt{N}} \int_0^{Nt} \varphi(\eta(s)) \, \mathrm{d}s.$$

Gaussian Hilbert space - an example

Example

Consider $\Omega_{ex} := \mathbb{R}$ *with* $\pi_{ex}(\mathrm{d}x) := \frac{1}{\sqrt{2\pi}} e^{-x^2/2} \mathrm{d}x$.
There is an orthogonal decomposition

$$L^2(\Omega_{ex}, \pi_{ex}) = \oplus_{n=0}^{\infty} \mathcal{H}_n^{ex}$$

where $\mathcal{H}_n^{ex}$ *contains polynomials of degree* n.

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where $\mathcal{H}_n^{ex}$ *contains polynomials of degree* n*. These are the Hermite polynomials, which can be constructed via Gram – Schmidt orthogonalization.*

Similarly with infinitely many variables, the same procedure gives

$$L^2(\Omega, \pi) = \oplus_{n=0}^{\infty} \mathcal{H}_n$$

where $\mathcal{H}_n$ is generated by the Wick polynomials of form $: \omega(x_1) \dots \omega(x_n) :$ with $x_1, \dots, x_n \in \mathbb{R}^d$, i.e. polynomials $\omega(x_1) \dots \omega(x_n)$ orthogonalized.

Sufficient conditions

- C. Kipnis, S. R. S. Varadhan, 1986 (reversible)
- B. Tóth, 1986 (non-reversible, discrete time)
- S. V. S. Varadhan, 1996: *(strong) sector condition*

$$\|S^{-1/2} A S^{-1/2}\| < \infty.$$

- S. Sethuraman, S. R. S. Varadhan, H-T. Yau, 2000: *graded/weak sector condition*: $\mathcal{L}^2(\Omega, \pi) = \oplus_{n=0}^{\infty} \mathcal{H}_n$ and
  - $S = \sum_n S_{n,n}$ with $S_{n,n} : \mathcal{H}_n \to \mathcal{H}_n$ and
  - $A = \sum_n A_{n,n+1} + A_{n,n-1}$ with $A_{n,n\pm 1} : \mathcal{H}_n \to \mathcal{H}_{n\pm 1}$

$$\left\| S_{n+1,n+1}^{-1/2} A_{n,n+1} S_{n,n}^{-1/2} \right\| \leq C n^{\gamma}$$

where $\gamma < 1$ or $\{\gamma = 1$ and C is small enough$\}$.

Modified graded sector condition

Setup: $\mathcal{L}^2(\Omega, \pi) = \oplus_{n=0}^{\infty} \mathcal{H}_n$ and

- $S = \sum_{n,j} S_{n,n+j}$ with $S_{n,n+j} : \mathcal{H}_n \to \mathcal{H}_{n+j}$, $|j| \leq 4$ even
- $A = \sum_n A_{n,n+1} + A_{n,n-1}$ with $A_{n,n\pm1} : \mathcal{H}_n \to \mathcal{H}_{n\pm1}$

Denote $D = \sum_n S_{n,n}$.

Lemma

- *upper bound on f*
- $0 \leq D \leq \gamma S$
- $\left\| S_{n+j,n+j}^{-1/2} A_{n,n+j} S_{n,n}^{-1/2} \right\| \leq c_2 n^{\beta}$ *with* $\beta < 1$ *or* $\beta = 1$, c_2 *small*
- $\left\| S_{n+j,n+j}^{-1/2} S_{n,n+j} S_{n,n}^{-1/2} \right\| \leq c_3 n^{\beta'}$ *with* $\beta' < 2$ *or* $\beta' = 2$, c_3 *small*
- $\left\| S_{n+j,n+j}^{-1/2} S_{n,n+j} S_{n,n}^{-1/2} \right\| \leq c_4 n^{\beta''}$ *with* $\beta'' < \infty$

The above conditions together are sufficient for the martingale approximation and the CLT.

The end

Thank you for your attention!