Efficient Valuation Methods for Contracts in Finance and Insurance

Kees Oosterlee [1,2]

[1]CWI, Center for Mathematics and Computer Science, Amsterdam,
[2]Delft University of Technology, Delft.

Joint Work with Fang Fang, Lech Grzelak, Stefan Singor

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Contents

- Option pricing method, based on Fourier-cosine expansions
  - Focus on European options and calibration
- Generalize to hybrid products
  - Models with stochastic interest rate; stochastic volatility

Financial industry; Banks at Work

- Pricing approach:
  1. Define some financial product
  2. Model asset prices involved (SDEs)
  3. Calibrate the model to market data (Numerics, Optimization)
  4. Model product price correspondingly (PDE, Integral)
  5. Price the product of interest (Numerics, MC)
  6. Set up hedge to remove the risk related to the product (Optimization)

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Pricing: Feynman-Kac Theorem

Given the final condition problem

$$\begin{cases} \dfrac{\partial v}{\partial t} & + \quad \dfrac{1}{2}\sigma^2 S^2 \dfrac{\partial^2 v}{\partial S^2} + rS\dfrac{\partial v}{\partial S} - rv = 0, \\ v(S,T) & = \quad \text{given} \end{cases}$$

Then the value, $v(S(t),t)$, is the unique solution of

$$v(S,t) = e^{-r(T-t)}\mathbb{E}^Q\{v(S(T),T)|S(t)\}$$

with the sum of the first derivatives of the option square integrable. and S satisfies the system of stochastic differential equations:

$$dS_t = rS_t dt + \sigma S_t dW_t^Q,$$

- Similar relations hold for other SDEs in Finance

Numerical Pricing Approach

- One can apply several numerical techniques to calculate the option price:
  - Numerical integration,
  - Monte Carlo simulation,
  - Numerical solution of the partial-(integro) differential equation (P(I)DE)
- Each of these methods has its merits and demerits.
- Numerical challenges:
  - Speed of solution methods (for example, for calibration)
  - Early exercise feature (P(I)DE $\longrightarrow$ free boundary problem)
  - The problem's dimensionality (not treated here)

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Motivation Fourier Methods

- Derive pricing methods that
  - are computationally fast
  - are not restricted to Gaussian-based models
  - should work as long as we have the characteristic function,

$$\phi(u) = \mathbb{E}\left(e^{iuX}\right) = \int_{-\infty}^{\infty} e^{iux} f(x) dx; \quad f(x) = \frac{1}{\pi}\int_0^{\infty} \text{Re}\,(\phi(u)e^{-iux}) du$$

    (available for Lévy processes and also for Heston's model).
  - In probability theory a characteristic function of a continuous random variable X, equals the Fourier transform of the density of X.
- Generalize basic method w.r.t. SDEs, contracts, applications

Class of Affine Jump Diffusion (AJD) processes

Duffie, Pan, Singleton (2000): The following system of SDEs:

$$d\mathbf{X}_t = \mu(\mathbf{X}_t)dt + \sigma(\mathbf{X}_t)d\mathbf{W}_t + d\mathbf{Z}_t,$$

is of the affine form, if the drift, volatility, jump intensity and interest rate satisfy:

$$\begin{aligned}
\mu(\mathbf{X}_t) &= a_0 + a_1\mathbf{X}_t \text{ for } (a_0, a_1) \in \mathbb{R}^n \times \mathbb{R}^{n\times n}, \\
\lambda(\mathbf{X}_t) &= b_0 + b_1^T\mathbf{X}_t, \text{ for } (b_0, b_1) \in \mathbb{R} \times \mathbb{R}^n, \\
\sigma(\mathbf{X}_t)\sigma(\mathbf{X}_t)^T &= (c_0)_{ij} + (c_1)_{ij}^T\mathbf{X}_t, (c_0, c_1) \in \mathbb{R}^{n\times n} \times \mathbb{R}^{n\times n\times n}, \\
r(\mathbf{X}_t) &= r_0 + r_1^T\mathbf{X}_t, \text{ for } (r_0, r_1) \in \mathbb{R} \times \mathbb{R}^n.
\end{aligned}$$

The discounted characteristic function then has the following form:

$$\phi(\mathbf{u}, \mathbf{X_t}, \mathbf{t}, \mathbf{T}) = e^{A(\mathbf{u},t,T)+\mathbf{B}(\mathbf{u},t,T)^T\mathbf{X_t}},$$

The coefficients $A(\mathbf{u}, t, T)$ and $\mathbf{B}(\mathbf{u}, \mathbf{t}, \mathbf{T})^\mathbf{T}$ satisfy a system of Riccati-type ODEs.

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The COS option pricing method, based on Fourier Cosine Expansions

Series Coefficients of the Density and the Ch.F.

- Fourier-Cosine expansion of a density function on interval $[a, b]$:

$$f(x) = {\sum_{n=0}^{\infty}}' F_n \cos\left(n\pi \frac{x-a}{b-a}\right),$$

with $x \in [a, b] \subset \mathbb{R}$ and the coefficients defined as

$$F_n := \frac{2}{b-a} \int_a^b f(x) \cos\left(n\pi \frac{x-a}{b-a}\right) dx.$$

- F_n has a direct relation to ch.f., $\phi(u) := \int_{\mathbb{R}} f(x) e^{iux} dx$ ($\int_{\mathbb{R}\setminus[a,b]} f(x) \approx 0$),

$$\begin{aligned} F_n \approx A_n \quad &:= \quad \frac{2}{b-a} \int_{\mathbb{R}} f(x) \cos\left(n\pi \frac{x-a}{b-a}\right) dx \\ &= \quad \frac{2}{b-a} \mathrm{Re}\left\{\phi\left(\frac{n\pi}{b-a}\right) \exp\left(-i\frac{na\pi}{b-a}\right)\right\}. \end{aligned}$$

Recovering Densities

- Replace F_n by A_n, and truncate the summation:

$$f(x) \approx \frac{2}{b-a} {\sum_{n=0}^{N-1}}' \mathrm{Re}\left\{ \phi\left(\frac{n\pi}{b-a}\right) \exp\left(in\pi\frac{-a}{b-a}\right)\right\} \cos\left(n\pi\frac{x-a}{b-a}\right),$$

- Example: $f(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2}$, $[a,b] = [-10, 10]$ and $x = \{-5, -4, \cdots, 4, 5\}$.

N	4	8	16	32	64
error	0.2538	0.1075	0.0072	4.04e-07	3.33e-16
cpu time (sec.)	0.0025	0.0028	0.0025	0.0031	0.0032

Exponential error convergence in N.

- Similar behaviour for other Lévy processes.

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Pricing European Options

- Start from the risk-neutral valuation formula:

$$v(x,t_0) = e^{-r\Delta t}\mathbb{E}^{\mathbb{Q}}\left[v(y,T)|x\right] = e^{-r\Delta t}\int_{\mathbb{R}} v(y,T)f(y|x)dy.$$

- Truncate the integration range:

$$v(x,t_0) = e^{-r\Delta t}\int_{[a,b]} v(y,T)f(y|x)dy + \varepsilon.$$

- Replace the density by the COS approximation, and interchange summation and integration:

$$\hat{v}(x,t_0) = e^{-r\Delta t}{\sum_{n=0}^{N-1}}{}' \mathrm{Re}\left\{\phi\left(\frac{n\pi}{b-a};x\right)e^{-in\pi\frac{a}{b-a}}\right\}V_n,$$

where the series coefficients of the payoff, V_n, are analytic.

Pricing European Options

- Log-asset prices: $x := \ln(S_0/K) \quad \text{and} \quad y := \ln(S_T/K)$,
- The payoff for European options reads

$$v(y,T) \equiv [\alpha \cdot K(e^y - 1)]^+.$$

- For a call option, we obtain

$$\begin{aligned} V_k^{call} &= \frac{2}{b-a}\int_0^b K(e^y - 1)\cos\left(k\pi\frac{y-a}{b-a}\right)dy \\ &= \frac{2}{b-a}K\left(\chi_k(0,b) - \psi_k(0,b)\right), \end{aligned}$$

- For a vanilla put, we find

$$V_k^{put} = \frac{2}{b-a}K\left(-\chi_k(a,0) + \psi_k(a,0)\right).$$

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Heston model

- The Heston stochastic volatility model can be expressed by the following 2D system of SDEs

$$\begin{cases} dS_t &= r_t S_t dt + \sqrt{\nu_t} S_t dW_t^S, \\ d\nu_t &= -\kappa(\nu_t - \overline{\nu})dt + \gamma\sqrt{\nu_t} dW_t^{\nu}, \end{cases}$$

- With $x_t = \log S_t$ this system is in the affine form.

$\Rightarrow$ Itô's Lemma: multi-D partial differential equation

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Characteristic Functions Heston Model

- For Lévy and Heston models, the ChF can be represented by

$$\begin{aligned} \phi(u;\mathbf{x}) &= \varphi_{levy}(u)\cdot e^{iu\mathbf{x}} \quad \text{with} \quad \varphi_{levy}(u) := \phi(u;0), \\ \phi(u;\mathbf{x},\nu_0) &= \varphi_{hes}(u;\nu_0)\cdot e^{iu\mathbf{x}}, \end{aligned}$$

- The ChF of the log-asset price for Heston's model:

$$\begin{aligned} \varphi_{hes}(u;\nu_0) &= \exp\left(iur\Delta t + \frac{\nu_0}{\gamma^2}\left(\frac{1-e^{-D\Delta t}}{1-Ge^{-D\Delta t}}\right)(\kappa - i\rho\gamma u - D)\right)\cdot \\ &\quad \exp\left(\frac{\kappa\bar{\nu}}{\gamma^2}\left(\Delta t(\kappa - i\rho\gamma u - D) - 2\log\left(\frac{1-Ge^{-D\Delta t}}{1-G}\right)\right)\right), \end{aligned}$$

with $D = \sqrt{(\kappa - i\rho\gamma u)^2 + (u^2 + iu)\gamma^2}$ and $G = \frac{\kappa - i\rho\gamma u - D}{\kappa - i\rho\gamma u + D}$.

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Heston Model

- We can present the V_k as $\mathbf{V}_k = U_k \mathbf{K}$, where

$$U_k = \begin{cases} \frac{2}{b-a}\left(\chi_k(0,b) - \psi_k(0,b)\right) & \text{for a call} \\ \frac{2}{b-a}\left(-\chi_k(a,0) + \psi_k(a,0)\right) & \text{for a put.} \end{cases}$$

- The pricing formula simplifies for Heston and Lévy processes:

$$v(\mathbf{x}, t_0) \approx \mathbf{K} e^{-r\Delta t} \cdot \mathrm{Re}\left\{ {\sum_{n=0}^{N-1}}' \varphi\left(\frac{n\pi}{b-a}\right) U_n \cdot e^{in\pi\frac{\mathbf{x}-a}{b-a}} \right\},$$

where $\varphi(u) := \phi(u; 0)$

Numerical Results

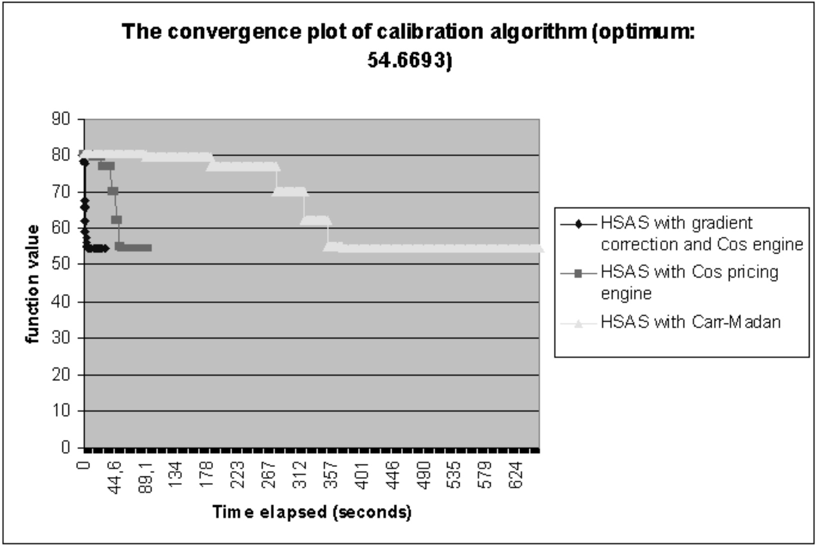
Pricing 21 strikes $K = 50, 55, 60, \cdots, 150$ simultaneously under Heston's model. Other parameters: $S_0 = 100, r = 0, q = 0, T = 1, \kappa = 1.5768, \gamma = 0.5751, \bar{\nu} = 0.0398, \nu_0 = 0.0175, \rho = -0.5711$.

	N	96	128	160
COS	(msec.)	2.039	2.641	3.220
	max. abs. err.	4.52e-04	2.61e-05	$4.40e-06$

Error analysis for the COS method is provided in the paper.

Numerical Results within Calibration

- Calibration for Heston's model: Around 10 times faster than Carr-Madan.



What do we do with the COS method?

- Generalizations:
  - Early-exercise options (Bermudan, barrier, American)
  - Context of CDS pricing (with Wim Schouten, Henrik Jönsson)
  - Swing options (commodity market)
  - Stochastic control problems, economic decision making (dikes, climate)
  - Asian options
  - Multi-asset options
- Generalize to hybrid products (Rabobank, Ortec Finance)
  - Models with stochastic interest rate; stochastic volatility
  - Heston Hull-White, Heston SV-LMM

An exotic contract: A hybrid product

- Based on sets of assets with different expected returns and risk levels.
- Proper construction may give reduced risk and an expected return greater than that of the least risky asset.
- A simple example is a portfolio with a stock with a high risk and return and a bond with a low risk and return.
- Example:

$$V(S, t_0) = \mathbb{E}^Q \left(e^{-\int_0^T r_s ds} \max\left(0, \frac{1}{2}\frac{S_T}{S_0} + \frac{1}{2}\frac{B_T}{B_0}\right)\right)$$

Heston-Hull-White hybrid model

- The Heston-Hull-White hybrid model can be expressed by the following 3D system of SDEs

$$\begin{cases} dS_t &= r_t S_t dt + \sqrt{\nu_t} S_t dW_t^S, \\ d\nu_t &= -\kappa(\nu_t - \overline{\nu})dt + \gamma\sqrt{\nu_t} dW_t^\nu, \\ dr_t &= \lambda\left(\theta_t - r_t\right) dt + \eta r_t^p dW_t^r, \end{cases}$$

- Full correlation matrix
- System is not in the affine form. The symmetric instantaneous covariance matrix is given by:

$$\begin{bmatrix} \nu_t & \rho_{x,\nu}\gamma\nu_t & \rho_{x,r}\eta r_t^p \sqrt{\nu_t} \\ * & \gamma^2\nu_t & \rho_{r,\nu}\gamma\eta r_t^p \sqrt{\nu_t} \\ * & * & \eta^2 r_t^{2p} \end{bmatrix}.$$

Linearization

⇒ By linearization of the non-affine terms in the covariance matrix, we find an approximation (set $p = 0$):

$$\begin{pmatrix} \nu_t & \rho_{x,\nu}\gamma\nu_t & \rho_{x,r}\eta\sqrt{\nu_t} \\ & \gamma^2\nu_t & \rho_{\nu,r}\eta\gamma\sqrt{\nu_t} \\ & & \eta^2 \end{pmatrix} \approx \underbrace{\begin{pmatrix} \nu_t & \rho_{x,\nu}\gamma\nu_t & \rho_{x,r}\eta\Psi_t \\ & \gamma^2\nu_t & \rho_{\nu,r}\eta\gamma\Psi_t \\ & & \eta^2 \end{pmatrix}}_{\mathbf{C}}.$$

⇒ We linearize the non-affine term $\sqrt{\nu_t}$ by Ψ_t:

$$\underbrace{\Psi_t = \mathbb{E}(\sqrt{\nu_t})}_{\text{analytic ChF}} \quad \text{or} \quad \Psi_t = \mathcal{N}\left(\mathbb{E}(\sqrt{\nu_t}), \mathbb{V}\text{ar}(\sqrt{\nu_t})\right).$$

⇒ The expectation for the CIR-type process is known analytically:

⇒ The model with the modified covariance structure, **C**, constitutes the affine version of the non-affine model.

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Reformulated HHW Model

- A well-defined Heston hybrid model with *indirectly imposed correlation*, $\rho_{x,r}$:

$$\begin{aligned}
dS_t &= r_t S_t dt + \sqrt{\nu_t} S_t dW_t^x + \Omega_t r_t^p S_t dW_t^r + \Delta \sqrt{\nu_t} S_t dW_t^\nu, \qquad S_0 > 0,\\
d\nu_t &= \kappa(\bar{\nu} - \nu_t)dt + \gamma \sqrt{\nu_t} dW_t^\nu, \qquad \nu_0 > 0,\\
dr_t &= \lambda(\theta_t - r_t)dt + \eta r_t^p dW_t^r, \qquad r_0 > 0,
\end{aligned}$$

with

$$\begin{aligned}
dW_t^x dW_t^\nu &= \hat{\rho}_{x,\nu},\\
dW_t^x dW_t^r &= 0,\\
dW_t^\nu dW_t^r &= 0,
\end{aligned}$$

- We have included a time-dependent function, Ω_t, and a parameter, Δ.

Basics

- Decompose a given general symmetric correlation matrix, $\mathbf{C}$, as $\mathbf{C} = \mathbf{L}\mathbf{L}^T$, where $\mathbf{L}$ is a lower triangular matrix with strictly positive entries.
- Rewrite a system of SDEs in terms of the independent Brownian motions with the help of the lower triangular matrix $\mathbf{L}$.

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"Equivalence"

- The HHW and HCIR models have $\rho_{r,\nu} = 0$, $\rho_{x,r} \neq 0$ and $\rho_{x,\nu} \neq 0$ and read: $d\mathbf{X}_t = [\ldots]dt +$

$$\begin{bmatrix} \rho_{x,r}\sqrt{\nu_t}S_t & \rho_{x,\nu}\sqrt{\nu_t}S_t & \sqrt{\nu_t}S_t\sqrt{1-\rho_{x,\nu}^2-\rho_{x,r}^2} \\ 0 & \gamma\sqrt{\nu_t} & 0 \\ \eta r_t^p & 0 & 0 \end{bmatrix} \begin{bmatrix} d\widetilde{W}_t^x \\ d\widetilde{W}_t^\nu \\ d\widetilde{W}_t^r \end{bmatrix}. \qquad (1)$$

- The reformulated hybrid model is given, in terms of the independent Brownian motions, by: $d\mathbf{X}_t = [\ldots]dt +$

$$\begin{bmatrix} \Omega_t r_t^p S_t & \sqrt{\nu_t}S_t\left(\hat{\rho}_{x,\nu}+\Delta\right) & \sqrt{\nu_t}S_t\sqrt{1-\hat{\rho}_{x,\nu}^2} \\ 0 & \gamma\sqrt{\nu_t} & 0 \\ \eta r_t^p & 0 & 0 \end{bmatrix} \begin{bmatrix} d\widetilde{W}_t^x \\ d\widetilde{W}_t^\nu \\ d\widetilde{W}_t^r \end{bmatrix},$$

"Equivalence"

- The reformulated HHW model is a well-defined Heston hybrid model with non-zero correlation, $\rho_{x,r}$, for:

$$\begin{aligned} \Omega_t = &\quad \rho_{x,r} r_t^{-p} \sqrt{\nu_t}, \\ \hat{\rho}^2_{x,\nu} = &\quad \rho^2_{x,\nu} + \rho^2_{x,r}, \\ \Delta = &\quad \rho_{x,\nu} - \hat{\rho}_{x,\nu}, \end{aligned}$$

- In order to satisfy the affinity constraints, we *approximate* Ω_t by a deterministic time-dependent function:

$$\Omega_t \approx \rho_{x,r} \mathbb{E}\left(r_t^{-p} \sqrt{\nu_t}\right) = \rho_{x,r} \mathbb{E}\left(r_t^{-p}\right) \mathbb{E}\left(\sqrt{\nu_t}\right),$$

  assuming independence between r_t and ν_t.

- The model is in the affine class
  $\Rightarrow$ Fast pricing of options with the COS method

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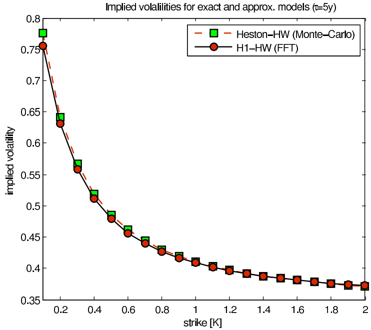
Numerical Experiment; Implied vol

- Implied volatilities for the HHW (obtained by Monte Carlo) and the approximate (obtained by COS) models.
- For short and long maturity experiments, we obtain a very good fit of the approximate to the full-scale HHW model.
- The parameters are $\theta = 0.03$, $\kappa = 1.2$, $\bar{\nu} = 0.08$, $\gamma = 0.09$, $\lambda = 1.1$, $\eta = 0.1$, $\rho_{x,v} = -0.7$, $\rho_{x,r} = 0.6$, $S_0 = 1$, $r_0 = 0.08$, $v_0 = 0.0625$, $a = 0.2813$, $b = -0.0311$ and $c = 1.1347$.



- $\tau = 5y$.

Other applications

- FX options (with Rabobank), although LMM are preferred for the IR modeling.
- Variable Annuities (with ING Insurance).
- Inflation options (with Ortec Finance).

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Inflation options: Efficient calibration of the inflation model

- A Heston type inflation model in combination with a Hull-White model for nominal and real interest rates, and nonzero correlations.
- An implied volatility skew/smile is present in inflation option market data.
- Complete risk-neutral inflation model under $\mathbb{Q}_n$:

$$\begin{cases} dI(t) = (r_n(t) - r_r(t))I(t)dt + \sqrt{\nu(t)}I(t)dW^I(t), \; I(0) \geq 0, \\ d\nu(t) = \kappa(\bar{\nu} - \nu(t))dt + \nu_v\sqrt{\nu(t)}dW^\nu(t), \; \nu(0) \geq 0, \end{cases}$$

with nominal and real interest rate processes given by:

$$\begin{cases} dr_n(t) = (\theta_n(t) - a_n r_n(t))dt + \eta_n dW^{r_n}(t), \; r_n(0) \geq 0, \\ dr_r(t) = (\theta_r(t) - \rho_{I,r}\eta_r\sqrt{\nu(t)} - a_r r_r(t))dt + \eta_r dW^{r_n}(t), \; r_r(0) \geq 0, \end{cases}$$

- Consumer Price Index I, variance process ν, and nominal and real interest rates, r_n and r_r.

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Inflation index and Year-on-Year options

- Inflation index options; call/put options written on the CPI:

$$M_n(t)\mathbb{E}^{\mathbb{Q}_n}\left[\frac{\max\left(\alpha(I(T)-K),0\right)}{M_n(T)}|\mathcal{F}_t\right] = P_n(t,T)\mathbb{E}^{\mathbb{Q}_n^T}\left[\max\left(\alpha(I_T(T)-K),0\right)|\mathcal{F}\right.$$

- Money savings account M_n, forward CPI $I_T(t) := I(t)P_r(t,T)/P_n(t,T)$.
- Year-on-year option: Series of forward starting call/put options written on the inflation rate.
- A cap protects the buyer from inflation above a certain rate (strike level). A floor gives downside protection. For $0 \leq t \leq T_1 \leq T_2$:

$$\begin{aligned} & M_n(t)\mathbb{E}^{\mathbb{Q}_n}\left[\frac{\max\left(\alpha(\frac{I(T_2)}{I(T_1)}-\tilde{K},0)\right)}{M_n(T_2)}|\mathcal{F}_t\right] \\ = \quad & P_n(t,T_2)\mathbb{E}^{T_2}\left[\max\left(\alpha\left(\frac{P_r(T_1,T_2)}{P_n(T_1,T_2)}\frac{I_{T_2}(T_2)}{I_{T_2}(T_1)}-\tilde{K}\right),0\right)|\mathcal{F}_t\right] \end{aligned}$$

Conclusions

- We presented the COS method, based on Fourier-cosine series expansions, for European options.
- The method also works efficiently for Bermudan and discretely monitored barrier options.
- COS method can be applied to affine approximations of HHW hybrid models.
- Generalized to full set of correlations, to Heston-CIR, and Heston-multi-factor models
- Papers available: http://ta.twi.tudelft.nl/mf/users/oosterle/oosterlee/ http://ta.twi.tudelft.nl/mf/users/oosterle/oosterlee/oosterleerecent.html

⇒ Top download in SIFIN !

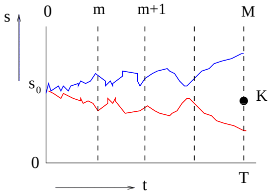
Summary

⇒ The linearization method provides a high quality approximation;

⇒ The projection procedure can be extended to high dimensions;

⇒ The method is straightforward, and does not involve complex techniques;

Pricing Bermudan Options



- The pricing formulae

$$\begin{cases} c(x,t_m) & = & e^{-r\Delta t}\int_{\mathbb{R}} v(y,t_{m+1})f(y|x)dy \\ v(x,t_m) & = & \max\left(g(x,t_m), c(x,t_m)\right) \end{cases}$$

and $v(x,t_0) = e^{-r\Delta t}\int_{\mathbb{R}} v(y,t_1)f(y|x)dy$.

  - Use Newton's method to locate the early exercise point x_m^*, which is the root of $g(x,t_m) - c(x,t_m) = 0$.
  - Recover $V_n(t_1)$ recursively from $V_n(t_M)$, $V_n(t_{M-1}), \cdots, V_n(t_2)$.
  - Use the COS formula for $v(x,t_0)$.

V_k-Coefficients

- Once we have x_m^*, we split the integral, which defines $V_k(t_m)$:

$$V_k(t_m) = \begin{cases} C_k(a, x_m^*, t_m) + G_k(x_m^*, b), & \text{for a call,} \\ G_k(a, x_m^*) + C_k(x_m^*, b, t_m), & \text{for a put,} \end{cases}$$

for $m = M-1, M-2, \cdots, 1.$ whereby

$$G_k(x_1, x_2) := \frac{2}{b-a} \int_{x_1}^{x_2} g(x, t_m) \cos\left(k\pi \frac{x-a}{b-a}\right) dx.$$

and

$$C_k(x_1, x_2, t_m) := \frac{2}{b-a} \int_{x_1}^{x_2} \hat{c}(x, t_m) \cos\left(k\pi \frac{x-a}{b-a}\right) dx.$$

Theorem

The $G_k(x_1, x_2)$ are known analytically and the $C_k(x_1, x_2, t_m)$ can be computed in $O(N \log_2(N))$ operations with the Fast Fourier Transform.

Bermudan Details

- Formula for the coefficients $C_k(x_1, x_2, t_m)$:

$$C_k(x_1, x_2, t_m) = e^{-r\Delta t}\mathrm{Re}\left\{\sum\nolimits_{j=0}^{\prime N-1} \varphi_{levy}\left(\frac{j\pi}{b-a}\right) V_j(t_{m+1}) \cdot M_{k,j}(x_1, x_2)\right\},$$

where the coefficients $M_{k,j}(x_1, x_2)$ are given by

$$M_{k,j}(x_1, x_2) := \frac{2}{b-a}\int_{x_1}^{x_2} e^{ij\pi\frac{x-a}{b-a}} \cos\left(k\pi\frac{x-a}{b-a}\right) dx,$$

- With fundamental calculus, we can rewrite $M_{k,j}$ as

$$M_{k,j}(x_1, x_2) = -\frac{i}{\pi}\left(M^c_{k,j}(x_1, x_2) + M^s_{k,j}(x_1, x_2)\right),$$

Hankel and Toeplitz

- Matrices $M_c = \{M^c_{k,j}(x_1, x_2)\}_{k,j=0}^{N-1}$ and $M_s = \{M^s_{k,j}(x_1, x_2)\}_{k,j=0}^{N-1}$ have special structure for which the FFT can be employed: M_c is a Hankel matrix,

$$M_c = \begin{bmatrix} m_0 & m_1 & m_2 & \cdots & m_{N-1} \\ m_1 & m_2 & \cdots & \cdots & m_N \\ \vdots & & & & \vdots \\ m_{N-2} & m_{N-1} & \cdots & & m_{2N-3} \\ m_{N-1} & \cdots & & m_{2N-3} & m_{2N-2} \end{bmatrix}_{N \times N}$$

and M_s is a Toeplitz matrix,

$$M_s = \begin{bmatrix} m_0 & m_1 & \cdots & m_{N-2} & m_{N-1} \\ m_{-1} & m_0 & m_1 & \cdots & m_{N-2} \\ \vdots & & \ddots & & \vdots \\ m_{2-N} & \cdots & m_{-1} & m_0 & m_1 \\ m_{1-N} & m_{2-N} & \cdots & m_{-1} & m_0 \end{bmatrix}_{N \times N}$$

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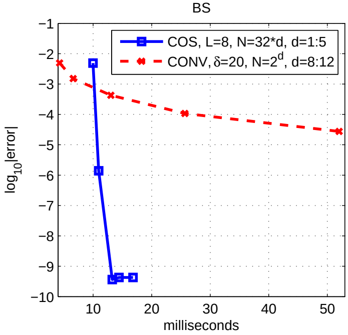
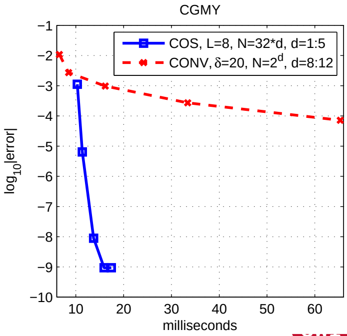
Bermudan puts with 10 early-exercise dates

Table: Test parameters for pricing Bermudan options

Test No.	Model	S_0	K	T	r	ν	Other Parameters
2	BS	100	110	1	0.1	0.2	–
3	CGMY	100	80	1	0.1	0	$C = 1, G = 5, M = 5, Y = 1.5$



(a) BS



(b) CGMY with $Y = 1.5$

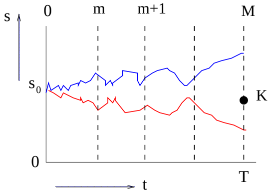
Heston Model

Defines the dynamics of (*log-stock*), x_t, and the variance, ν_t:

$$\begin{aligned} dx_t &= \left(\mu - \frac{1}{2}\nu_t\right) dt + \rho\sqrt{\nu_t}dW_{1,t} + \sqrt{1-\rho^2}\sqrt{\nu_t}dW_{2,t} \\ d\nu_t &= \kappa\left(\bar{\nu} - \nu_t\right) dt + \gamma\sqrt{\nu_t}dW_{1,t}, \end{aligned}$$

- $W_{1,t}$ and $W_{2,t}$ are independent; ρ is the correlation between the log-stock and the variance processes.
- The Feller condition, $2\kappa\bar{\nu} \geq \gamma^2$, guarantees that ν_t stays positive.

Bermudan Options



- Based on backward recursion. The continuation value is given by

$$c(x_m, \nu_m, t_m) = e^{-r\Delta t}\mathbb{E}^{\mathbb{Q}}_{t_m}\left[v(x_{m+1}, \nu_{m+1}, t_{m+1})\right],$$

which can be written as:

$$c(x_m, \nu_m, t_m) = e^{-r\Delta t} \cdot \int_{\mathbb{R}}\int_{\mathbb{R}} v(x_{m+1}, \nu_{m+1}, t_{m+1}) p_{x,\nu}(x_t, \nu_t | x_s, \nu_s) dx_{m+1} d\nu_{m+1}.$$

followed by $v(x, t_m) = \max\left(g(x_m, t_m), c(x_m, \nu_m, t_m)\right)$.

- *Scaled* log-asset price: $x_m = \ln\left(S_m / K\right)$.

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Joint Distribution of Log-Stock and Log-Variance

- For path-dependent options, we need the joint distribution $p_{x,\nu}(x_t, \nu_t | x_s, \nu_s)$ with $0 < s < t$ (log-stock and log-variance processes, given the information at the current time):

$$p_{x,\nu}(x_t, \nu_t | x_s, \nu_s) = p_{x|\nu}(x_t | \nu_t, x_s, \nu_s) \cdot p_\nu(\nu_t | \nu_s),$$

- $p_{x|\nu}$: density of the log-stock process, given the variance value.

$\Rightarrow$ Relevant information in the Fourier domain.

The Left-Side Tail

- With $q := 2\kappa\bar{v}/\gamma^2 - 1, \text{ and } \zeta := 2\kappa/\left((1-e^{-\kappa(t-s)})\gamma^2\right)$, $I_q(\cdot)$ the modified, order q, Bessel function of the first kind, the density of ν_t given ν_s reads:

$$p_\nu\left(\nu_t|\nu_s\right) = \zeta e^{-\zeta(\nu_s e^{-\kappa(t-s)}+\nu_t)}\left(\frac{\nu_t}{\nu_s e^{-\kappa(t-s)}}\right)^{\frac{q}{2}} I_q\left(2\zeta e^{-\frac{1}{2}\kappa(t-s)}\sqrt{\nu_s\nu_t}\right).$$

- The left-side tail is characterized by $q \in [-1, \infty)$. With $\kappa \geq 0$, $\bar{v} \geq 0$ and $\gamma \geq 0$, a near-singular problem occurs when $q \in [-1, 0]$.

Transformation to Log-Variance Process

- The density of the log-variance process reads:

$$p_{\ln(\nu)}\left(\sigma_t|\sigma_s\right) =$$

$$\zeta e^{-\zeta\left(e^{\sigma_s}e^{-\kappa(t-s)}+e^{\sigma_t}\right)}\left(\frac{e^{\sigma_t}}{e^{\sigma_s}e^{-\kappa(t-s)}}\right)^{\frac{q}{2}} e^{\sigma_t} I_q\left(2\zeta e^{-\frac{1}{2}\kappa(t-s)}\sqrt{e^{\sigma_s}e^{\sigma_t}}\right),$$

where $\sigma_s := \ln(\nu_s)$ and $p_{\ln(\nu)}(\sigma_t|\sigma_s)$ denotes the density of the log-variance, given the information at current time.

Joint Density

- We have $p_{x,\ln(\nu)}(x_t, \sigma_t | x_s, \sigma_s) = p_{x|\ln(\nu)}(x_t | \sigma_t, x_s, \sigma_s) \cdot p_{\ln(\nu)}(\sigma_t | \sigma_s)$, with $p_{x|\ln(\nu)}$ the probability density of log-stock at a future time.
  There is no closed-form expression for $p_{x|\ln(\nu)}$, but one can derive its conditional characteristic function, $\hat{\varphi}(\omega; x_s, \sigma_t, \sigma_s)$,

$$
\begin{aligned}
\hat{\varphi}(\omega; x_s, \sigma_t, \sigma_s) &:= \mathbb{E}_s\left[\exp\left(i\omega x_t | \sigma_t\right)\right] \\
&= \exp\left(i\omega\left[x_s + \mu(t-s) + \frac{\rho}{\gamma}\left(e^{\sigma_t} - e^{\sigma_s} - \kappa\bar{\nu}(t-s)\right)\right]\right) \cdot \\
&\quad \Phi\left(\omega\left(\frac{\kappa\rho}{\gamma} - \frac{1}{2}\right) + \frac{1}{2}i\omega^2(1-\rho^2); e^{\sigma_t}, e^{\sigma_s}\right),
\end{aligned}
$$

  where $\Phi(u; \nu_t, \nu_s)$ is the ChF of the time-integrated variance.

Heston Model

- The ChF, $\Phi(v; \nu_t, \nu_s)$, reads [Broadie-Kaya 2004]:

$$
\begin{aligned}
\Phi(\upsilon; \nu_t, \nu_s) &:= \mathbb{E}\left[\exp\left(i\upsilon \int_s^t \nu_\tau d\tau\right) \middle| \nu_t, \nu_s\right] \\
&= \frac{I_q\left[\sqrt{\nu_t \nu_s}\frac{4\gamma(\upsilon)e^{-\frac{1}{2}\gamma(\upsilon)(t-s)}}{\gamma^2(1-e^{-\gamma(\upsilon)(t-s)})}\right]}{I_q\left[\sqrt{\nu_t \nu_s}\frac{4\kappa e^{-\frac{1}{2}\kappa(t-s)}}{\gamma^2(1-e^{-\kappa(t-s)})}\right]} \cdot \\
&\quad \frac{\gamma(\upsilon)e^{-\frac{1}{2}(\gamma(\upsilon)-\kappa)(t-s)}(1-e^{-\kappa(t-s)})}{\kappa(1-e^{-\gamma(\upsilon)(t-s)})} \cdot \\
&\quad \exp\left(\frac{\nu_s+\nu_t}{\gamma^2}\left[\frac{\kappa(1+e^{-\kappa(t-s)})}{1-e^{-\kappa(t-s)}} - \frac{\gamma(\upsilon)(1+e^{-\gamma(\upsilon)(t-s)})}{1-e^{-\gamma(\upsilon)(t-s)}}\right]\right),
\end{aligned}
$$

with $\gamma(\upsilon) := \sqrt{\kappa^2 - 2i\gamma^2\upsilon}$.

Density Recovery by Fourier Cosine Expansions

- Apply the COS method to approximate the conditional probability density, $p_{x|\ln(\nu)}$.

$$p_{x|\ln(\nu)}(x_{m+1}|\sigma_{m+1},x_m,\sigma_m) = {\sum_{n=0}^{\infty}}' P_n(\sigma_{m+1},x_m,\sigma_m)\cos\left(n\pi\frac{x_{m+1}-a}{b-a}\right).$$

Coefficients P_n have a direct relation to the characteristic function and are therefore known, i.e.

$$P_n(\sigma_{m+1},x_m,\sigma_m) \approx \frac{2}{b-a}\mathrm{Re}\left\{\hat{\varphi}\left(\frac{n\pi}{b-a};x_m,\sigma_{m+1},\sigma_m\right)\ e^{-in\pi\frac{a}{b-a}}\right\},$$

with $\hat{\varphi}(\theta;x,\sigma_{m+1},\sigma_m)$ given earlier.

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Quadrature Rule in Log-Variance Dimension

- After truncating the integration region by $[a_\nu, b_\nu] \times [a, b]$, we compute

$$c_1(x_m, \sigma_m, t_m) := e^{-r\Delta t} \cdot \int_{a_\nu}^{b_\nu} \left[\int_a^b v(x_{m+1}, \sigma_{m+1}, t_{m+1}) p_{x|\ln(\nu)}\left(x_{m+1}| \, \sigma_{m+1}, x_m, \sigma_m\right) dx_{m+1} \right] p_{\ln(\nu)}\left(\sigma_{m+1}|\sigma_m\right) d\sigma_{m+1}.$$

- Apply J-point quadrature integration rule (like Gauss-Legendre quadrature, composite Trapezoidal rule, etc.) to the outer integral:

$$c_2(x_m, \sigma_m, t_m) := e^{-r\Delta t} \sum_{j=0}^{J-1} w_j \cdot p_{\ln(\nu)}(\varsigma_j|\sigma_m) \cdot \left[\int_a^b v(x_{m+1}, \varsigma_j, t_{m+1}) p_{x|\ln(\nu)}\left(x_{m+1}| \, \varsigma_j, x_m, \sigma_m\right) dx_{m+1} \right].$$

- A Gauss-Legendre rule gives exponential error convergence for smooth functions, such as $p_{\ln(\nu)}$,

COS Reconstruction in Log-Stock Dimension

- Replace $p_{x|\ln(\nu)}$, by the COS approximation, and interchange summation over n and integration over x_{m+1}:

$$c_3(x_m, \sigma_m) := e^{-r\Delta t} \sum_{j=0}^{J-1} w_j {\sum_{n=0}^{N-1}}' V_{n,j}(t_{m+1}) \mathrm{Re}\left\{ \tilde{\varphi}\left(\frac{n\pi}{b-a}, \varsigma_j, \sigma_m\right) e^{in\pi \frac{x_m - a}{b-a}} \right\},$$

with

$$V_{n,j}\left(t_{m+1}\right) := \frac{2}{b-a} \int_a^b v(x_{m+1}, \varsigma_j, t_{m+1}) \cos\left(n\pi \frac{x_{m+1} - a}{b-a}\right) dx_{m+1},$$

and

$$\tilde{\varphi}(\omega, \sigma_{m+1}, \sigma_m) := p_{\ln(\nu)}(\sigma_{m+1}|\sigma_m) \cdot \varphi\left(\omega; 0, e^{\sigma_{m+1}}, e^{\sigma_m}\right).$$

- Kernel $\tilde{\varphi}$ characterizes the Heston model.
- The Bessel function present in $p_{\ln(\nu)}$ cancels with a Bessel function in the denominator of φ, leaving one Bessel-term.

COS Reconstruction in Log-Stock Dimension

- With early-exercise points, $x^*(\sigma_m, t_m)$, determined, recursion can be used to compute the Bermudan option price:
  - At t_M: $v(x_M, \sigma_M, t_M) = g(x_M);$
  - At t_m, with $m = 1, 2, \cdots, M-1$:

$$\hat{v}(x_m, \sigma_m, t_m) = \begin{cases} g(x_m) & \text{for } x \in [a, x^*(\sigma_m, t_m)] \\ c_3(x_m, \sigma_m, m) & \text{for } x \in (x^*(\sigma_m, t_m), b] \end{cases} \quad (2)$$

    for a put option.
  - At t_0: $\hat{v}(x_0, \sigma_0, t_0) = c_3(x_0, \sigma_0, t_0).$
- By backward recursion, the cosine coefficients of $\hat{v}(x_1, \sigma_1, t_1)$ can be recovered with the FFT, from those of $\hat{v}(x_M, \sigma_M, t_M)$ in $O((M-1)JN\,\ell)$ operations, with $\ell = \max[\log_2(N), J]$.

$\Rightarrow$ As with the COS method for Bermudan options under Lévy processes

European Test Results

- Test No.1 ($q = 0.6$): $\gamma = 0.5, \kappa = 5, \bar{\nu} = 0.04, T = 1$;
- Other parameters to determine the values of the *put* include $\rho = -0.9, \nu_0 = 0.04, S_0 = 100, K = 100, r = 0$.
- Convergence in J for Test No.1 ($q = 0.6$) with $N = 2^7, M = 12$ and the European option reference value is 7.5789038982.

($J = 2^d$)	Cosine expansion plus Gauss-Legendre Rule			
	TOL $= 10^{-6}$		TOL $= 10^{-8}$	
d	time(sec)	error	time(sec)	error
4	0.12	1.02 10^{-2}	0.12	1.41
5	0.42	-1.85 10^{-5}	0.40	2.99 10^{-5}
6	1.59	-1.54 10^{-5}	1.54	-6.41 10^{-6}
7	7.07	-1.34 10^{-5}	6.49	-6.32 10^{-7}

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Numerical Results $q < 0$

- Convergence in J as $q \to -1$;
- Test No.2 ($q = -0.84$): $\gamma = 0.5, \kappa = 0.5, \bar{\nu} = 0.04$, $T = 1$;
- Test No.3 ($q = -0.96$): $\gamma = 1, \kappa = 0.5, \bar{\nu} = 0.04$, $T = 10$.
- Fourier cosine expansion plus Gauss-Legendre rule, $N = 2^8$, $M = 12$, TOL$= 10^{-7}$,
- European reference values are 6.2710582179 (Test No. 2) and 13.0842710701 (Test No.3).

$(J = 2^d)$	Test No. 2 ($q = -0.84$)				Test No. 3 ($q = -0.96$)			
	time(sec)				time(sec)			
d	total	Init.	Loop	error	total	Init.	Loop	error
6	3.03	2.85	0.18	5.63	3.11	2.93	0.18	-22.7
7	13.3	12.78	0.56	6.89 10^{-3}	12.1	11.55	0.53	-8.51 10^{-2}
8	56.4	52.32	4.07	-2.12 10^{-5}	55.7	51.74	4.00	-1.60 10^{-3}

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Bermudan Option Result

- A negative correlation coefficient, ρ, is often observed in market data.
- Test No. 4 ($q = -0.47$): $S_0 = \{90, 100, 110\}, K = 100, T = 0.25, r = 0.04, \kappa = 1.15, \gamma = 0.39, \rho = -0.64, \bar{\nu} = 0.0348, \nu_0 = 0.0348$.

	S_0			time (sec)		
M	90	100	110	total	Init.	Loop
20	9.9783714	3.2047434	0.9273568	68.9	58.2	10.7
40	9.9916484	3.2073345	0.9281068	81.9	59.3	22.6
60	9.9957789	3.2079202	0.9280425	93.2	59.4	33.8

Conclusions

- Bermudan options under Heston's model with a Fourier-based method.
- The near-singular problem in the left-side tail of the variance density has been dealt with by a change of variables to the log-variance domain.
- Pricing formula is derived by applying a Fourier series expansion technique to the log-stock and a quadrature rule to the log-variance dimension.
- With the Feller condition satisfied, we get highly accurate prices within a fraction of a second.
- The challenge is to price options for the Feller condition not satisfied. Choosing 128 points in both dimensions is usually sufficient for an error reduction of the order 10^{-4}.
- The computation of the Bessel functions in the initialization step of the algorithm dominates the overall computation time in that case.

Truncation Range $[a_\nu, b_\nu]$ for Log-variance Density

- Use Newton's method to determine the interval boundaries, according to a pre-defined error tolerance, $p_{\ln(\nu)}(x|\sigma_0; T) <$TOL for $x \in \mathbb{R}\backslash[a_\nu, b_\nu]$.
- The derivative of $p_{\ln(\nu)}(\sigma_t|\sigma_s)$ w.r.t. σ_t with Maple:

$$\frac{dp_{\ln(\nu)}(\sigma_t|\sigma_s)}{d\sigma_t} = -\left[\left(-\zeta e^{\sigma_t} - q - 1\right) I_q\left(2\sqrt{\zeta e^{\sigma_t} u}\right) - I_{q+1}\left(2\sqrt{\zeta e^{\sigma_t} u}\right)\right] \cdot$$

$$\zeta e^{-u-\zeta e^{\sigma_t}+\sigma_t} \cdot \left(\frac{\zeta e^{\sigma_t}}{u}\right)^{q/2},$$

with $u := \zeta e^{\sigma_s - \kappa(t-s)}$.

- *Initial guess:* We estimate the center by the logarithm of the mean value of the variance

$$\ln(\mathbb{E}(\nu_t)) = \ln\left(\nu_0 e^{-\kappa T} + \bar{\nu}\left(1 - e^{-\kappa T}\right)\right).$$

- As the left tail usually decays much slower than the right tail and the *speed of decay* seems closely related to the value of q, we use:

$$[a_\nu^0, b_\nu^0] = \left[\ln(\mathbb{E}(\nu_t)) - \frac{5}{1+q}, \ \ln(\mathbb{E}(\nu_t)) + \frac{2}{1+q}\right]$$

V_k-Coefficients

- Once we have x_m^*, we split the integral, which defines $V_k(t_m)$:

$$V_k(t_m) = \begin{cases} C_k(a, x_m^*, t_m) + G_k(x_m^*, b), & \text{for a call,} \\ G_k(a, x_m^*) + C_k(x_m^*, b, t_m), & \text{for a put,} \end{cases}$$

for $m = M-1, M-2, \cdots, 1$. whereby

$$G_k(x_1, x_2) := \frac{2}{b-a} \int_{x_1}^{x_2} g(x, t_m) \cos\left(k\pi \frac{x-a}{b-a}\right) dx.$$

and

$$C_k(x_1, x_2, t_m) := \frac{2}{b-a} \int_{x_1}^{x_2} \hat{c}(x, t_m) \cos\left(k\pi \frac{x-a}{b-a}\right) dx.$$

Theorem

The $G_k(x_1, x_2)$ are known analytically and the $C_k(x_1, x_2, t_m)$ can be computed in $O(N \log_2(N))$ operations with the Fast Fourier Transform.

Bermudan Details

- Formula for the coefficients $C_k(x_1, x_2, t_m)$:

$$C_k(x_1, x_2, t_m) = e^{-r\Delta t}\mathrm{Re}\left\{{\sum_{j=0}^{N-1}}' \varphi_{levy}\left(\frac{j\pi}{b-a}\right) V_j(t_{m+1}) \cdot M_{k,j}(x_1, x_2)\right\},$$

  where the coefficients $M_{k,j}(x_1, x_2)$ are given by

$$M_{k,j}(x_1, x_2) := \frac{2}{b-a}\int_{x_1}^{x_2} e^{ij\pi\frac{x-a}{b-a}} \cos\left(k\pi\frac{x-a}{b-a}\right) dx,$$

- With fundamental calculus, we can rewrite $M_{k,j}$ as

$$M_{k,j}(x_1, x_2) = -\frac{i}{\pi}\left(M^c_{k,j}(x_1, x_2) + M^s_{k,j}(x_1, x_2)\right),$$

Hankel and Toeplitz

- Matrices $M_c = \{M^c_{k,j}(x_1, x_2)\}_{k,j=0}^{N-1}$ and $M_s = \{M^s_{k,j}(x_1, x_2)\}_{k,j=0}^{N-1}$ have special structure for which the FFT can be employed: M_c is a Hankel matrix,

$$M_c = \begin{bmatrix} m_0 & m_1 & m_2 & \cdots & m_{N-1} \\ m_1 & m_2 & \cdots & \cdots & m_N \\ \vdots & & & & \vdots \\ m_{N-2} & m_{N-1} & \cdots & & m_{2N-3} \\ m_{N-1} & \cdots & & m_{2N-3} & m_{2N-2} \end{bmatrix}_{N\times N}$$

and M_s is a Toeplitz matrix,

$$M_s = \begin{bmatrix} m_0 & m_1 & \cdots & m_{N-2} & m_{N-1} \\ m_{-1} & m_0 & m_1 & \cdots & m_{N-2} \\ \vdots & & \ddots & & \vdots \\ m_{2-N} & \cdots & m_{-1} & m_0 & m_1 \\ m_{1-N} & m_{2-N} & \cdots & m_{-1} & m_0 \end{bmatrix}_{N\times N}$$

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Bermudan puts with 10 early-exercise dates

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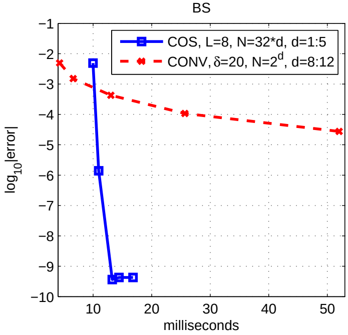
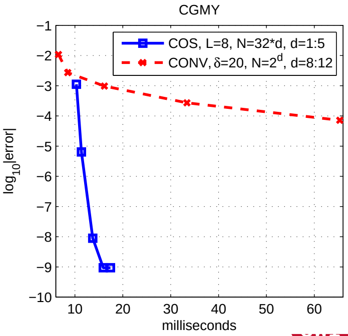


(c) BS



(d) CGMY with $Y = 1.5$

Pricing Discrete Barrier Options

- The price of an M-times monitored up-and-out option satisfies

$$\begin{cases} c(x, t_{m-1}) & = & e^{-r(t_m - t_{m-1})} \int_{\mathbb{R}} v(x, t_m) f(y|x) dy \\ v(x, t_{m-1}) & = & \begin{cases} e^{-r(T-t_{m-1})} Rb, & x \geq h \\ c(x, t_{m-1}), & x < h \end{cases} \end{cases}$$

where $h = \ln(H/K)$, and $v(x, t_0) = e^{-r(t_m - t_{m-1})} \int_{\mathbb{R}} v(x, t_1) f(y|x) dy$.

- The technique:
  - Recover $V_n(t_1)$ recursively, from $V_n(t_M)$, $V_n(t_{M-1}), \cdots, V_n(t_2)$ in $O((M-1)N \log_2(N))$ operations.
  - Split the integration range at the barrier level (no Newton required)
  - Insert $V_n(t_1)$ in the COS formula to get $v(x, t_0)$, in $O(N)$ operations.

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Monthly-monitored Barrier Options

Table: Test parameters for pricing barrier options

Test No.	Model	S_0	K	T	r	q	Other Parameters
1	NIG	100	100	1	0.05	0.02	$\alpha = 15, \beta = -5, \delta = 0.5$

Option Type	Ref. Val.	N	time (milli-sec.)	error
DOP	2.139931117	2^7	3.7	1.28e-3
		2^8	5.4	4.65e-5
		2^9	8.4	1.39e-7
		2^{10}	14.7	1.38e-12
DOC	8.983106036	2^7	3.7	1.09e-3
		2^8	5.3	3.99e-5
		2^9	8.3	9.47e-8
		2^{10}	14.8	5.61e-13

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Conclusions

- The COS method is highly efficient for density recovery, for pricing European, Bermudan and discretely -monitored barrier options
- Convergence is exponential, usually with small N