Föllmer-Schweizer or Galtchouck-Kunita-Watanabe Decomposition?
A Comparison and Description

Tahir Choulli, Nele Vandaele, Michèle Vanmaele

Workshop on Actuarial and Financial Statistics
August 29-30, 2011
Hedging problem

- Financial product $P(t, S_t)$
- Depends on a risky asset $S$

A bank sells an option and wants to replicate its payoff $P(T, S_T)$ by trading in stocks (liquid assets).

Hedging strategy $\varphi = (\xi, \eta)$

Investment in risky asset and cash in order to reduce the risk related to a financial product.

- Hedging portfolio $V_t = \xi_t S_t + \eta_t$
- Cost process

\[ C = V - \int \xi dS = V - \xi \cdot S \]
**Hedging problem**

- Financial product $P(t, S_t)$
- Depends on a risky asset $S$

A bank sells an option and wants to replicate its payoff $P(T, S_T)$ by trading in stocks (liquid assets).

**Hedging strategy $\varphi = (\xi, \eta)$**

Investment in risky asset and cash in order to reduce the risk related to a financial product.

- Hedging portfolio $V_t = \xi_t S_t + \eta_t$
- Cost process

\[
C = V - \int \xi dS = V - \xi \cdot S
\]
Hedging problem

- Financial product $P(t, S_t)$
- Depends on a risky asset $S$

A bank sells an option and wants to replicate its payoff $P(T, S_T)$ by trading in stocks (liquid assets).

**Hedging strategy** $\varphi = (\xi, \eta)$

Investment in risky asset and cash in order to reduce the risk related to a financial product.

- Hedging portfolio $V_t = \xi_t S_t + \eta_t$
- Cost process

\[
C = V - \int \xi dS = V - \xi \cdot S
\]
Hedging in Black-Scholes model

- $dS_t = \sigma S_t dW_t$ (martingale measure $Q$, no interest rate)
- Perfect replication by self-financing strategies
- Martingale representation

$$P(T, S_T) = E^Q[P(T, S_T)] + \int_0^T Z_t dW_t = E^Q[P(T, S_T)] + \int_0^T \xi_t dS_t$$

Where in case of a European option

$$\xi_t = \frac{\partial P(t, S_t)}{\partial s} \text{ with } P(t, s) = E^Q[P(T, S_T)|S_t = s]$$

delta-hedge
Hedging in Black-Scholes model

- \( dS_t = \sigma S_t dW_t \) (martingale measure \( Q \), no interest rate)
- perfect replication by self-financing strategies
- martingale representation

\[
P(T, S_T) = E^Q[P(T, S_T)] + \int_0^T Z_t dW_t = E^Q[P(T, S_T)] + \int_0^T \xi_t dS_t
\]

- where in case of a European option

\[
\xi_t = \frac{\partial P(t, S_t)}{\partial s} \text{ with } P(t, s) = E^Q[P(T, S_T)|S_t = s]
\]
delta-hedge
Hedging in Black-Scholes model

- $dS_t = \sigma S_t dW_t$ (martingale measure $Q$, no interest rate)
- perfect replication by self-financing strategies
- martingale representation

$$P(T, S_T) = E^Q[P(T, S_T)] + \int_0^T Z_t dW_t = E^Q[P(T, S_T)] + \int_0^T \xi_t dS_t$$

- where in case of a European option

$$\xi_t = \left. \frac{\partial P(t, S_t)}{\partial s} \right|_{S_t = s} \text{ with } P(t, s) = E^Q[P(T, S_T)|S_t = s]$$

delta-hedge
Hedging in Black-Scholes model

- $dS_t = \sigma S_t dW_t$ (martingale measure $Q$, no interest rate)
- perfect replication by self-financing strategies
- martingale representation

$$P(T, S_T) = E^Q[P(T, S_T)] + \int_0^T Z_t dW_t = E^Q[P(T, S_T)] + \int_0^T \xi_t dS_t$$

where in case of a European option

$$\xi_t = \frac{\partial P(t, S_t)}{\partial s} \quad \text{with} \quad P(t, s) = E^Q[P(T, S_T)|S_t = s]$$

delta-hedge
jumps, stochastic volatility or trading constraints

- martingale representation above does not hold
- ‘every claim attainable and replicated by self-financing strategy’ is not valid
- relax one of these two conditions
- hedging is an approximation problem
utility maximization: non-linear pricing/hedging rule

\[ \max_{\xi} E \left[ U(c + \int_0^T \xi_t dS_t - H) \right] \]

quadratic hedging: linear pricing/hedging rule

\[ \min_{\xi} E \left[ (c + \int_0^T \xi_t dS_t - H)^2 \right] \quad \text{(mean-variance)} \]
\[ \min_{\xi} E \left[ (C_T - C_t)^2 \mid \mathcal{F}_t \right] \quad \text{((local) risk minimization)} \]

optimal hedging portfolio (if exists) is \( L^2 \)-projection of \( H \) onto the (linear) subspace of hedgeable claims
Outline

1 Introduction

2 Quadratic hedging
   ■ Risk-minimization
   ■ GKW-decomposition
   ■ Local risk-minimization
   ■ Föllmer-Schweizer decomposition

3 GKW- versus FS-decomposition

4 (Counter)examples

5 References
finding optimal hedging portfolio ⇔ finding GKW-decomposition or finding FS-decomposition

- Martingale case: easy to determine $\xi$ which is same for RM and MVH ($\eta$ differs)
- Semimartingale case = martingale + drift
  - LRM: general solution
  - MVH: no general solution due to self-financing condition
finding optimal hedging portfolio ⇔ finding GKW-decomposition or finding FS-decomposition

Martingale case: easy to determine $\xi$ which is same for RM and MVH ($\eta$ differs)

Semimartingale case = martingale + drift
- LRM: general solution
- MVH: no general solution due to self-financing condition
Risk-minimization

- \( S \): local \textbf{martingale} under measure \( P \)
- \( T \)-contingent claim \( H \in L^2(P) \)
- not self-financing strategy but mean self-financing strategy, i.e. cost process is martingale
- \( H \)-admissible strategy: value process has terminal value \( H \)
- value process \( V \) of discounted portfolio:

\[
V_t = E[H | \mathcal{F}_t]
\]
Föllmer and Sondermann (1986): solution to risk-minimization problem can be found by

\[ H = E[H] + \int_0^T \xi_u dS_u + L_T \]

with \( L \) local martingale orthogonal to \( S \)

- by martingale property

\[ V_t = E[H|\mathcal{F}_t] = E[H] + \int_0^t \xi_u dS_u + L_t \]

Hedging strategy: \( \varphi = (\xi_t, V_t - \xi_t S_t) \)
Föllmer and Sondermann (1986): solution to risk-minimization problem can be found by Galtchouk-Kunita-Watanabe decomposition:

\[ H = E[H] + \int_0^T \xi_u dS_u + L_T \]

with \( L \) local martingale orthogonal to \( S \).

- by martingale property:

\[ V_t = E[H|\mathcal{F}_t] = E[H] + \int_0^t \xi_u dS_u + L_t \]

Hedging strategy: \( \varphi = (\xi_t, V_t - \xi_t S_t) \)
Föllmer and Sondermann (1986): solution to risk-minimization problem can be found by

\[ H = \mathbb{E}[H] + \int_0^T \xi_u dS_u + L_T \]

with \( L \) local martingale orthogonal to \( S \)

by martingale property

\[ V_t = \mathbb{E}[H|\mathcal{F}_t] = \mathbb{E}[H] + \int_0^t \xi_u dS_u + L_t \]

Hedging strategy: \( \varphi = (\xi_t, V_t - \xi_t S_t) \)
Föllmer and Sondermann (1986): solution to risk-minimization problem can be found by

\[
H = E[H] + \int_0^T \xi_u dS_u + L_T
\]

with \( L \) local martingale orthogonal to \( S \)

- by martingale property

\[
V_t = E[H | \mathcal{F}_t] = E[H] + \int_0^t \xi_u dS_u + L_t
\]

- Hedging strategy: \( \varphi = (\xi_t, V_t - \xi_t S_t) \)
Orthogonal

$X$ is orthogonal to $Y$ ($X \perp Y$) $\iff$ $[X, Y]$ is a local martingale

with $[X, Y] = XY - Y \cdot X - X \cdot Y = XY - \int YdX - \int XdY$

$\Rightarrow$ compensator of $[X, Y]$: $\langle X, Y \rangle = 0$

**Remark:** $\langle \cdot, \cdot \rangle$ is measure dependent!

use $\langle \cdot, \cdot \rangle$ to determine $\xi$ from GKW-decomposition
Orthogonal

$X$ is orthogonal to $Y$ ($X \perp Y$) $\iff [X, Y]$ is a local martingale

with $[X, Y] = XY - Y \cdot X - X \cdot Y = XY - \int YdX - \int XdY$

$\Rightarrow$ compensator of $[X, Y]: \langle X, Y \rangle = 0$

Remark: $\langle \cdot, \cdot \rangle$ is measure dependent!

use $\langle \cdot, \cdot \rangle$ to determine $\xi$ from GKW-decomposition
computation of $\xi$ from GKW-decomposition:

\[
dV_t = \xi_t dS_t + dL_t
\]

\[
d\langle V, S \rangle_t = \xi_t d\langle S, S \rangle_t + d\langle L, S \rangle_t \quad \Leftrightarrow \quad \xi_t = \frac{d\langle V, S \rangle_t}{d\langle S, S \rangle_t}
\]

risk process and remaining risk:

\[
R_t(\varphi) = E[(C_T(\varphi) - C_t(\varphi))^2 | \mathcal{F}_t] = E[(L_T - L_t)^2 | \mathcal{F}_t]
\]
GKW-decomposition

- computation of $\xi$ from GKW-decomposition:

\[ dV_t = \xi_t dS_t + dL_t \]

\[ d\langle V, S \rangle_t = \xi_t d\langle S, S \rangle_t + d\langle L, S \rangle_t \quad \iff \quad \xi_t = \frac{d\langle V, S \rangle_t}{d\langle S, S \rangle_t} \]

- risk process and remaining risk:

\[ R_t(\varphi) = E[(C_T(\varphi) - C_t(\varphi))^2|F_t] = E[(L_T - L_t)^2|F_t] \]
Local risk-minimization

- $S = S_0 + M + B$: one-dimensional $P$-semimartingale with
  - $M$: square-integrable local martingale, $M_0 = 0$
  - $B$: predictable process with finite variation

- Not possible to find a risk-minimizing strategy (Schweizer (1988))
$S = S_0 + M + B$: one-dimensional $P$-semimartingale with
- $M$: square-integrable local martingale, $M_0 = 0$
- $B$: predictable process with finite variation

Not possible to find a risk-minimizing strategy (Schweizer (1988))

- minimization of the risk $R_t(\varphi) = E[(C_T(\varphi) - C_t(\varphi))^2|\mathcal{F}_t]$ replaced by

**New criterion**

$$\liminf_{n \to \infty} r^{T_n}(\varphi, \Delta) \geq 0$$

with

$$r^\tau[\varphi, \Delta](\omega, t) := \sum_{t_i, t_{i+1} \in \tau} \frac{R_{t_i}(\varphi + \Delta|_{(t_i, t_{i+1})}) - R_{t_i}(\varphi)}{E[\langle M \rangle_{t_{i+1}} - \langle M \rangle_{t_i}|\mathcal{F}_{t_i}]} 1_{(t_i, t_{i+1})}(t)$$

- riskiness of cost process measured locally in time
Schweizer (1991, 2008): $H$-admissible strategy $\varphi$ is LRM strategy iff $\varphi$ is mean-self-financing and martingale $C(\varphi)$ is orthogonal to martingale part $M$ of price process $S$.

under assumptions
(A1) $\langle M \rangle$ is $P$-a.s. strictly increasing on $[0, T]$
(A2) $B$ is continuous
(A3) $B$ is absolutely continuous w.r.t. $\langle M \rangle$ with density $\lambda$ satisfying $E[\langle \int \lambda dM \rangle] < \infty$

LRM strategy $\varphi$ follows from FS-decomposition of $H \in L^2(P)$
FS-decomposition

**Föllmer-Schweizer decomposition**

\[ H = H_0 + \int_0^T \xi_u^{FS} dS_u + L_T^{FS} \]

with \( L_T^{FS} \) local martingale orthogonal to \( M \)

- **How?** Recall that \( C = V - \xi \cdot S \) is \( P \)-martingale and \( P \)-orthogonal to \( M \)
- **Answer:** define equivalent martingale measure \( Q \) such that \( C \) is also \( Q \)-martingale

**Minimal martingale measure** \( Q \) related to a \( P \)-semimartingale

The martingale measure such that any local martingale orthogonal to \( M \) under \( P \) remains local martingale under \( Q \)
Föllmer-Schweizer decomposition

\[ H = H_0 + \int_0^T \xi_u^{FS} dS_u + L_T^{FS} \]

with \( L_T^{FS} \) local martingale orthogonal to \( M \)

- How? Recall that \( C = V - \xi \cdot S \) is \( P \)-martingale and \( P \)-orthogonal to \( M \)
- Answer: define equivalent martingale measure \( Q \) such that \( C \) is also \( Q \)-martingale

**Minimal martingale measure** \( Q \) related to a \( P \)-semimartingale

The martingale measure such that any local martingale orthogonal to \( M \) under \( P \) remains local martingale under \( Q \)
LRM strategy \( \varphi \) is given by

\[
\varphi_t = \left( \xi_t^{\text{FS}}, V_t - \xi_t^{\text{FS}} S_t \right)
\]

where

\[
V_t = E^Q[H|\mathcal{F}_t] = E^Q[H] + \int_0^t \xi_u^{\text{FS}} dS_u + L_t^{\text{FS}}
\]
computation of $\xi^{FS}$ from FS-decomposition:

$$dV_t^M + dV_t^B = dV_t = \xi^F_t dS_t + dL_t^{FS} = \xi^F_t (dM_t + dB_t) + dL_t^{FS}$$

$$d\langle V^M, M \rangle_t = \xi^F_t d\langle M, M \rangle_t + \underbrace{d\langle L^{FS}, M \rangle_t}_{=0}$$

$$\Leftrightarrow \xi^F_t = \frac{d\langle V^M, M \rangle_t}{d\langle M, M \rangle_t}$$
computation of $\xi^{FS}$ from FS-decomposition:

$$dV_t^M + dV_t^B = dV_t = \xi^{FS}_t dS_t + dL^{FS}_t = \xi^{FS}_t (dM_t + dB_t) + dL^{FS}_t$$

$$d\langle V^M, M \rangle_t = \xi^{FS}_t d\langle M, M \rangle_t + d\langle L^{FS}, M \rangle_t = 0$$

$$\Leftrightarrow \quad \xi^{FS}_t = \frac{d\langle V^M, M \rangle_t}{d\langle M, M \rangle_t}$$
Outline

1. Introduction
2. Quadratic hedging
3. GKW- versus FS-decomposition
   - Continuous case
   - Discontinuous case
4. (Counter)examples
5. References
GKW versus FS

- GKW-decomposition

\[ E^Q[H|\mathcal{F}_t] = E^Q[H] + (\xi \cdot S)_t + L_t \quad \text{with} \quad \langle L, S \rangle^Q = 0 \]

\( L \) is \( Q \)-local martingale

- FS-decomposition

\[ E^Q[H|\mathcal{F}_t] = E^Q[H] + (\xi^{FS} \cdot S)_t + L_{FS}^t \quad \text{with} \quad \langle L^{FS}, M \rangle = 0 \]

\( L^{FS} \) is \( P \)-local martingale

- \( L^{FS} \) is \( Q \)-local martingale by definition of MMM.

Question 1: Is \( L^{FS} \) orthogonal to \( S \), i.e. \( \langle L^{FS}, S \rangle^Q = 0 \)?

Question 2: Is \( L \) \( P \)-martingale orthogonal to \( M \)?
GKW versus FS

- **GKW-decomposition**

\[
E^Q[H|\mathcal{F}_t] = E^Q[H] + (\xi \cdot S)_t + L_t \quad \text{with} \quad \langle L, S \rangle^Q = 0
\]

$L$ is $Q$-local martingale

- **FS-decomposition**

\[
E^Q[H|\mathcal{F}_t] = E^Q[H] + (\xi^{FS} \cdot S)_t + L^{FS}_t \quad \text{with} \quad \langle L^{FS}, M \rangle = 0
\]

$L^{FS}$ is $P$-local martingale

- $L^{FS}$ is $Q$-local martingale by definition of MMM.

**Question 1:** Is $L^{FS}$ orthogonal to $S$, i.e. $\langle L^{FS}, S \rangle^Q = 0$?

**Question 2:** Is $L$ $P$-martingale orthogonal to $M$?
GKW versus FS

- \( L^{FS} = L + (\xi - \xi^{FS}) \cdot S \)
- relation between
  \[
  \xi = \frac{d\langle V, S \rangle^Q}{d\langle S, S \rangle^Q} \quad \text{and} \quad \xi^{FS} = \frac{d\langle V^M, M \rangle}{d\langle M, M \rangle}
  \]
  from GKW-decomposition: \( dV_t = \xi_t dS_t + L_t \)

- \[
  \xi^{FS} = \xi + \frac{d\langle L, M \rangle}{d\langle M, M \rangle}
  \]
GKW versus FS

- \( L_{FS} = L + (\xi - \xi_{FS}) \cdot S \)
- relation between

\[
\xi = \frac{d\langle V, S \rangle^Q}{d\langle S, S \rangle^Q} \quad \text{and} \quad \xi_{FS} = \frac{d\langle V^M, M \rangle}{d\langle M, M \rangle}?
\]

- from GKW-decomposition: \( dV_t = \xi_t dS_t + L_t \)

\[ d\langle V^M, M \rangle = \xi d\langle M, M \rangle + \langle L, M \rangle \]

- \( \xi_{FS} = \xi + \frac{d\langle L, M \rangle}{d\langle M, M \rangle} \)
GKW versus FS

- \( L^{FS} = L + (\xi - \xi^{FS}) \cdot S \)

- relation between

\[
\xi = \frac{d\langle V, S \rangle^Q}{d\langle S, S \rangle^Q} \quad \text{and} \quad \xi^{FS} = \frac{d\langle V^M, M \rangle}{d\langle M, M \rangle}?
\]

- from GKW-decomposition: \( dV_t = \xi_t dS_t + L_t \)

\[
\frac{d\langle V^M, M \rangle}{d\langle M, M \rangle} = \xi \frac{d\langle M, M \rangle}{d\langle M, M \rangle} + \frac{d\langle L, M \rangle}{d\langle M, M \rangle}
\]

\[
\xi^{FS} = \xi + \frac{d\langle L, M \rangle}{d\langle M, M \rangle}
\]
GKW versus FS

\[ L^{FS} = L + (\xi - \xi^{FS}) \cdot S \]

relation between

\[ \xi = \frac{d\langle V, S \rangle^Q}{d\langle S, S \rangle^Q} \quad \text{and} \quad \xi^{FS} = \frac{d\langle V^M, M \rangle}{d\langle M, M \rangle} \]

from GKW-decomposition: \( dV_t = \xi_t dS_t + L_t \)

\[ \frac{d\langle V^M, M \rangle}{d\langle M, M \rangle} = \xi \frac{d\langle M, M \rangle}{d\langle M, M \rangle} + \frac{d\langle L, M \rangle}{d\langle M, M \rangle} \]

\[ \xi^{FS} = \xi + \frac{d\langle L, M \rangle}{d\langle M, M \rangle} \]
\[ \langle L^{FS}, S \rangle^Q = \langle L, S \rangle^Q + (\xi - \xi^{FS}) \cdot \langle S, S \rangle^Q = 0 \iff \xi = \xi^{FS} \]

\[ \xi^{FS} = \xi \iff \langle L, M \rangle = 0 \]
\[ \iff L \text{ is } P\text{-martingale orthogonal to } M \]

\[ \text{Question 1 } \iff \text{Question 2 } \iff \text{GKW and FS coincide under MMM} \]
\[ \langle L^{FS}, S \rangle^Q = \langle L, S \rangle^Q + (\xi - \xi^{FS}) \cdot \langle S, S \rangle^Q = 0 \iff \xi = \xi^{FS} \]

\[ \xi^{FS} = \xi \iff \langle L, M \rangle = 0 \]
\[ \iff \text{ } L \text{ is } \mathcal{P}\text{-martingale orthogonal to } M \]

- Question 1 \iff Question 2 \iff GKW and FS coincide under MMM
\[ \langle L^{FS}, S \rangle^Q = \langle L, S \rangle^Q + (\xi - \xi^{FS}) \cdot \langle S, S \rangle^Q = 0 \iff \xi = \xi^{FS} \]

\[ \xi^{FS} = \xi \iff \langle L, M \rangle = 0 \]
\[ \iff L \text{ is } P\text{-martingale orthogonal to } M \]

- Question 1 \iff Question 2 \iff GKW and FS coincide under MMM
Continuous case

- $S$ is **continuous** process $\Rightarrow$ GKW and FS coincide under MMM

  Föllmer & Schweizer (1991) proved preservation of orthogonality
  
  answer to question 1: $\langle L^{FS}, S \rangle^Q = 0$

- $M$ is also continuous

  $\Rightarrow \langle L, M \rangle = [L, M] = [L, S] = \langle L, S \rangle^Q = 0$

  $\Leftrightarrow \xi^{FS} = \xi \Leftrightarrow L^{FS} = L$

  $\text{GKW} = \text{FS}$
Continuous case

- $S$ is **continuous** process $\Rightarrow$ GKW and FS coincide under MMM

  Föllmer & Schweizer (1991) proved preservation of orthogonality

  answer to question 1: $\langle L^{FS}, S \rangle^Q = 0$

- $M$ is also continuous

  $\Rightarrow \langle L, M \rangle = [L, M] = [L, S] = \langle L, S \rangle^Q = 0$

  $\Leftrightarrow \xi^{FS} = \xi \Leftrightarrow L^{FS} = L$

  **GKW = FS**
Continuous case

Original measure $P$

\[ \frac{dQ}{dP} \]

Minimal martingale measure $Q$

Föllmer-Schweizer decomposition

\[ (\xi, \eta) \]

Galtchouk-Kunita-Watanabe decomposition

Michèle Vanmaele — Föllmer-Schweizer or Galtchouck-Kunita-Watanabe Decomposition? A Comparison and Description
Discontinuous case

- **S** is **discontinuous** process $\Rightarrow$ GKW and FS DO NOT coincide under MMM
  - orthogonality not preserved from $P$ to $Q$ or vice versa

\[
\langle L, S \rangle^Q = 0 \not\Rightarrow \langle L, M \rangle = 0
\]

Is $Q$-local martingale $L$ a $P$-local martingale?

- question of orthogonality by definition formulated as question of being martingale:

\[
\langle L, M \rangle = 0 \iff [L, M] \text{ is } P\text{-local martingale}
\]

\[
\text{GKW} \neq \text{FS}
\]
Discontinuous case

- $S$ is discontinuous process $\Rightarrow$ GKW and FS DO NOT coincide under MMM
- orthogonality not preserved from $P$ to $Q$ or vice versa

$$\langle L, S \rangle^Q = 0 \not\Rightarrow \langle L, M \rangle = 0$$

Is $Q$-local martingale $L$ a $P$-local martingale?

- question of orthogonality by definition formulated as question of being martingale:

$$\langle L, M \rangle = 0 \iff [L, M] \text{ is } P\text{-local martingale}$$

GKW $\neq$ FS
Discontinuous case

- $S$ is *discontinuous* process $\Rightarrow$ GKW and FS do not coincide under MMM
- Orthogonality not preserved from $P$ to $Q$ or vice versa

\[
\langle L, S \rangle^Q = 0 \nRightarrow \langle L, M \rangle = 0
\]

Is $Q$-local martingale $L$ a $P$-local martingale?

- Question of orthogonality by definition formulated as question of being martingale:

\[
\langle L, M \rangle = 0 \iff [L, M] \text{ is } P\text{-local martingale}
\]

**GKW $\neq$ FS**
in terms of predictable characteristics an additional condition different from orthogonality condition on $Q$-local martingale to be a $P$-local martingale is required

**Main proposition (Choulli, Vandaele, V)**

$L$ $Q$-local martingale, then $L$ is $P$-local martingale if and only if

$$\lambda'_t c_t \beta_t + \int [\lambda'_t x - \lambda'_t \Delta \langle M \rangle_t \lambda_t] W_t(x) F_t(dx) = 0.$$

**Note:** Use uniqueness of the representation theorem
\[ \xi^{FS} \text{ in terms of } \xi \text{ (Choulli, Vandaele, V)} \]

\( (\tilde{\beta}, \tilde{f}, \tilde{g}, L^\perp) \): quadruplet associated with \( L \) under \( Q \), then

\[ \xi^{FS} - \xi = \tilde{\Phi} \quad L^{FS} = L - \tilde{\Phi} \cdot S, \]

with

\[ \tilde{\Phi} := \Sigma^{inv} \int x\tilde{f}(x)\left[\lambda'x - \lambda'\Delta \langle M \rangle \lambda \right] F(dx), \]

and \( \Sigma^{inv} \) is the Moore-Penrose pseudoinverse of \( \Sigma \)

\[ \Sigma := c + \int xx' F(dx) \]
Predictable characteristics (Choulli, Vandaele, V)

Consider a square-integrable $\mathcal{F}_T$-measurable random variable $H$, and denote by $(H_0, \xi^{FS}, L^{FS})$ its FS-decomposition components. Then the following holds

$$\xi^{FS} = \Sigma^{inv} \left\{ c\tilde{\phi} + \int x\tilde{f}(x)F(dx) \right\} \quad \text{and} \quad L^{FS} = V - \xi^{FS} \cdot S. $$

Here $(\tilde{\phi}, \tilde{f}, \tilde{g}, \tilde{K}^\perp)$: quadruplet associated with $V^M$, and $\Sigma$ is a random symmetric matrix given by

$$\Sigma := c + \int xx'F(dx).$$
Practical example

- $S = S_0 + S^c + x \star (\mu - \nu) + B$ with $\mu$ a random measure and $\nu$ its $P$-compensator, $dB_t = bdt$

- $H$ is contingent claim

- $V_t = E^Q[H | \mathcal{F}_t] = f(t, S_t)$ with $f$ a $C^{1,2}$-function

- Itô-formula

\[
V_t = V_0 + \int_0^t f_x(s, S_{s-})dS + \int_0^t [f_t(s, S) + \frac{1}{2}f_{xx}(s, S_{s-})]ds \\
+ \sum_{0 < s \leq t} [f(s, S_{s-}) - f(s, S_{s-}) - f_x(s, S_{s-})\Delta S_s]
\]
Practical example

- **P-martingale part of \( V \):**

\[
V^M = f_x(\cdot, S_-) \cdot S^c + [f(\cdot, S_- + x) - f(\cdot, S_-)] \ast (\mu - \nu)
\]

- **FS-decomposition of \( H \)**

\[
\xi_{FS} = \left[ c f_x(\cdot, S_-) + \int_{\mathbb{R}} x[f(\cdot, S_- + x) - f(\cdot, S_-)] F(dx) \right] / \left[ c + \int_{\mathbb{R}} x^2 F(dx) \right]
\]

\[
L_{FS} = V - V_0 - \xi_{FS} \cdot S
\]
Countereexample

- one-dimensional discounted process modelled as Lévy process:

\[
S_t := S_0 \mathbb{E}(S)_t, \quad \bar{S}_t := \sigma W_t + \gamma \tilde{p}_t + \mu t
\]

- \( p \): standard Poisson process with intensity 1
- \( \tilde{p}_t = p_t - t, \ 0 \leq t \leq T \): compensated Poisson process
- \( W \): standard Brownian motion
- \( S_0 > 0, \sigma > 0, \gamma > -1, \ 0 \neq \mu \gamma < \sigma^2 + \gamma^2 \)
- decomposition of \( S \): \( S = S_0 + M + B \)

\[
dS_t = S_{t-}d\bar{S}_t
\]
Counterexample

- decomposition of $S$: $S = S_0 + M + B$?

$$dM_t = S_t - (\sigma dW_t + \gamma d\tilde{p}_t), \quad dB_t = \mu S_t dt$$

- minimal martingale measure $Q$? density given by

$$Z := \mathcal{E}(-\lambda \cdot M) \text{ with } dB = \lambda d\langle M \rangle_N$$

- for this model

$$\lambda_t = \frac{1}{S_t} \frac{\mu}{\sigma^2 + \gamma^2}, \quad N_t = \sigma_1 W_t + \gamma_1 \tilde{p}_t, \quad \sigma_1 := \frac{-\mu \sigma}{\sigma^2 + \gamma^2}, \quad \gamma_1 := \frac{-\mu \gamma}{\sigma^2 + \gamma^2}$$
Counterexample

- decomposition of $S$: $S = S_0 + M + B$?

$$
\begin{align*}
    dM_t &= S_{t-}(\sigma dW_t + \gamma d\tilde{\rho}_t), \\
    dB_t &= \mu S_{t-}dt
\end{align*}
$$

- minimal martingale measure $Q$? density given by

$$
Z := \mathcal{E}(-\lambda \cdot M) \text{ with } dB = \lambda d\langle M \rangle
$$

- for this model

$$
\begin{align*}
    \lambda_t &= \frac{1}{S_{t-}} \frac{\mu}{\sigma^2 + \gamma^2}, \\
    N_t &= \sigma_1 W_t + \gamma_1 \tilde{\rho}_t, \\
    \sigma_1 &= \frac{-\mu \sigma}{\sigma^2 + \gamma^2}, \\
    \gamma_1 &= \frac{-\mu \gamma}{\sigma^2 + \gamma^2}
\end{align*}
$$
Counterexample

- European put option with payoff: \( H = (K - S_T)_+ \)
- by independent increments of \( S \)

\[
V_t = f(t, S_t) \quad \text{with} \quad f(t, x) = E^Q \left[ (K - x\frac{S_T}{S_t})_+ \right]
\]

- distribution function of \( S \)?

\[
S_t = S_0 \mathcal{E}(\mathcal{S})_t = S_0 e^{-\mathcal{S}_t - \mathcal{S}_0 - \frac{1}{2} \langle \mathcal{S}^c \rangle_t} \prod_{s \leq t} (1 + \Delta \mathcal{S}_s) e^{-\Delta \mathcal{S}_s}
\]

with \( \langle \mathcal{S}^c \rangle_t = \langle \sigma W \rangle_t = \sigma^2 t \)

\( \Delta \mathcal{S}_s = \gamma \Delta \tilde{p}_s = \gamma \Delta p_s \) being zero or one

\[
S_t = S_0 e^{\sigma W_t + \tilde{p}_t \log(1+\gamma) + (\mu - \frac{1}{2} \sigma^2 + \log(1+\gamma) - \gamma) t}
\]
strictly increasing distribution function in $y$, $y = \log x$

$$F(s, y) = Q\left(\frac{S_s}{S_0} \leq x\right) = Q\left(\log(S_s) - \log(S_0) \leq y\right)$$

$$= Q(\sigma W_s + \log(1 + \gamma)\tilde{p}_s + \bar{\mu}s \leq y)$$

by stationarity property of $S$, for $x > 0$:

$$f(t, x) = E^Q \left[ (K - x \frac{S_T}{S_t})_+ \right] = xE^Q \left[ (\frac{K}{x} - \mathcal{E}(\overline{S})_{T-t})_+ \right]$$

$$= x \int_{-\infty}^{\log \frac{K}{x}} (\frac{K}{x} - e^y) dF(T - t, y)$$

$$= KF(T - t, \log \frac{K}{x}) - x \int_{-\infty}^{\log \frac{K}{x}} e^y F_y(T - t, y) dy$$
Counterexample

- \( f \in C^{1,2}((0, T) \times (0, +\infty)) \), apply Itô to \( f(t, S_t) \) and \( V \) is \( Q \)-martingale

\[
V_t = V_0 + \int_0^t f_x(u, S_{u-}) dS_u + (\Gamma \cdot \tilde{p}^Q)_t
\]

\[
\tilde{p}^Q := p_t - (1 + \gamma_1)t
\]

\[
\Gamma_u := f(u, S_{u-}(1 + \gamma)) - f(u, S_{u-}) - f_x(u, S_u)\gamma S_{u-}
\]

- \( \xi \) from GKW-decomposition

\[
\xi = \frac{d\langle V, S \rangle^Q}{d\langle S, S \rangle^Q} = f_x + \Gamma \frac{d\langle \tilde{p}^Q, S \rangle^Q}{d\langle S, S \rangle^Q}
\]
GKW-decomposition: \( V = V_0 + \xi \cdot S + L \)

- \( \xi \) given by

\[
\xi_t = f_x(t, S_{t-}) + \Gamma_t \frac{d\langle \tilde{p}^Q, S \rangle_t^Q}{d\langle S, S \rangle_t^Q}
\]

\[
= f_x(t, S_{t-}) + \frac{\Gamma_t}{S_{t-}} \frac{\gamma(1 + \gamma_1)}{\sigma^2 + \gamma^2(1 + \gamma_1)}
\]

- \( V_t = V_0 + \int_0^t f_x(u, S_{u-})dS_u + (\Gamma \cdot \tilde{p}^Q)_t \)

\[
L = \Gamma \cdot \tilde{p}^Q - \Gamma \frac{\gamma(1 + \gamma_1)}{S_{t-} \sigma^2 + \gamma^2(1 + \gamma_1)} \cdot S
\]
The FS-decomposition of $H$ and the GKW-decomposition under $Q$ of $V$ differ.

**Proof**

- difference between $\xi^{\text{FS}}$ and $\xi$:

  $$\xi^{\text{FS}} - \xi = \frac{d\langle L, M \rangle}{d\langle M, M \rangle} = \frac{\Gamma}{S_-} \frac{\mu \gamma^2 \sigma^2}{(\sigma^2 + \gamma^2)^2(\sigma^2 + \gamma^2(1 + \gamma_1))}$$

- compute $\Gamma_t = f(t, S_{t-}(1 + \gamma)) - f(t, S_t) - f_x(t, S_t)\gamma S_{t-}$

  $$f_x(t, x) = -\int_\infty^{\log \frac{K}{x}} e^y F_y(T - t, y) \, dy$$
Proof (continued)

- plug in expressions for $f$ and $f_x$:
  \[
  \Gamma_t = f(t, S_{t^-}(1 + \gamma)) - f(t, S_{t^-}) - f_x(t, S_t)\gamma S_{t^-}
  \]
  \[
  = \int_{s_1(t)}^{s_2(t)} [K - S_{t^-}(1 + \gamma)e^y]F_y(T - t, y)dy
  \]
  with $s_1(t) := \log \frac{K}{S_{t^-}}$ and $s_2(t) := s_1(t) - \log(1 + \gamma)$

- $\Gamma \neq 0$ for $\gamma \neq 0$:
  
  $(-1 <) \gamma < 0$: $s_1 < s_2$ and $[K - S_{t^-}(1 + \gamma)e^y]F_y(T - t, y) > 0$
  
  $\gamma > 0$: $s_1 > s_2$ and
  
  \[
  \Gamma_t = \int_{s_2(t)}^{s_1(t)} [-K + S_{t^-}(1 + \gamma)e^y]F_y(T - t, y)dy > 0
  \]
Counterexample

Proof (continued)

- plug in expressions for $f$ and $f_x$:

$$\Gamma_t = f(t, S_{t-}(1 + \gamma)) - f(t, S_{t-}) - f_x(t, S_t)\gamma S_{t-}$$

$$= \int_{s_1(t)}^{s_2(t)} [K - S_{t-}(1 + \gamma)e^y]F_y(T - t, y)dy$$

with $s_1(t) := \log \frac{K}{S_{t-}}$ and $s_2(t) := s_1(t) - \log(1 + \gamma)$

- $\Gamma \neq 0$ for $\gamma \neq 0$:
  
  $(-1 <) \gamma < 0$: $s_1 < s_2$ and $[K - S_{t-}(1 + \gamma)e^y]F_y(T - t, y) > 0$

  $\gamma > 0$: $s_1 > s_2$ and

  $$\Gamma_t = \int_{s_2(t)}^{s_1(t)} [-K + S_{t-}(1 + \gamma)e^y]F_y(T - t, y)dy > 0$$
Proof (continued)

- plug in expressions for $f$ and $f_x$:

$$
\Gamma_t = f(t, S_t(1 + \gamma)) - f(t, S_t) - f_x(t, S_t)\gamma S_t
= \int_{s_1(t)}^{s_2(t)} \left[K - S_t(1 + \gamma)e^y\right]F_y(T - t, y)dy
$$

with $s_1(t) := \log \frac{K}{S_t}$ and $s_2(t) := s_1(t) - \log(1 + \gamma)$

- $\Gamma \neq 0$ for $\gamma \neq 0$:
  
  $(-1 <)\gamma < 0$: $s_1 < s_2$ and $\left[K - S_t(1 + \gamma)e^y\right]F_y(T - t, y) > 0$
  
  $\gamma > 0$: $s_1 > s_2$ and

$$
\Gamma_t = \int_{s_2(t)}^{s_1(t)} \left[-K + S_t(1 + \gamma)e^y\right]F_y(T - t, y)dy > 0
$$
Outline

1 Introduction

2 Quadratic hedging

3 GKW- versus FS-decomposition

4 (Counter)examples

5 References
T. Choulli, N. Vandaele, and M. Vanmaele.
The Föllmer-Schweizer decomposition: Comparison and description. 

H. Föllmer, M. Schweizer.
Hedging of contingent claims under incomplete information. 

H. Föllmer, D. Sondermann.
Hedging of non-redundant contingent claims. 

M. Schweizer.
Hedging of options in a general semimartingale model. 

M. Schweizer.
Risk-minimality and orthogonality of martingales. 

M. Schweizer.
Option hedging for semimartingales. 

M. Schweizer.
Local risk minimization for multidimensional assets and payment streams. 
Thank you for your attention

This study was supported by a grant of Research Foundation-Flanders