UNIVERSITEIT GENT

Föllmer-Schweizer or Galtchouck-Kunita-Watanabe Decomposition? A Comparison and Description

Tahir Choulli, Nele Vandaele, Michèle Vanmaele

Workshop on Actuarial and Financial Statistics

August 29-30, *2011*

Outline

UNIVERSITEIT GENT

T. Choulli, N. Vandaele, and M. Vanmaele.
The Föllmer-Schweizer decomposition: Comparison and description.
Stochastic Processes and their Applications, 120(6):853–872, 2010.

UNIVERSITEIT GENT

Outline

UNIVERSITEIT GENT

Hedging problem

- Financial product $P(t, S_t)$
- Depends on a risky asset S

A bank sells an option and wants to replicate its payoff $P(T, S_T)$ by trading in stocks (liquid assets).

Hedging strategy $\varphi = (\xi, \eta)$

Investment in risky asset and cash in order to reduce the risk related to a financial product.

- Hedging portfolio $V_t = \xi_t S_t + \eta_t$
- Cost process

$$C = V - \int \xi dS = V - \xi \cdot S$$

UNIVERSITEIT GENT

Hedging problem

- Financial product $P(t, S_t)$
- Depends on a risky asset S

A bank sells an option and wants to replicate its payoff $P(T, S_T)$ by trading in stocks (liquid assets).

Hedging strategy $\varphi = (\xi, \eta)$

Investment in risky asset and cash in order to reduce the risk related to a financial product.

- Hedging portfolio $V_t = \xi_t S_t + \eta_t$
- Cost process

$$C = V - \int \xi dS = V - \xi \cdot S$$

Hedging problem

- Financial product $P(t, S_t)$
- Depends on a risky asset S

A bank sells an option and wants to replicate its payoff $P(T, S_T)$ by trading in stocks (liquid assets).

Hedging strategy $\varphi = (\xi, \eta)$

Investment in risky asset and cash in order to reduce the risk related to a financial product.

- Hedging portfolio $V_t = \xi_t S_t + \eta_t$
- Cost process

$$C = V - \int \xi dS = V - \xi \cdot S$$

UNIVERSITEIT GENT

Complete market

Hedging in Black-Scholes model

- $dS_t = \sigma S_t dW_t$ (martingale measure Q, no interest rate)
- perfect replication by self-financing strategies
- martingale representation

$$P(T,S_T) = E^Q[P(T,S_T)] + \int_0^T Z_t dW_t = E^Q[P(T,S_T)] + \int_0^T \xi_t dS_t$$

- where in case of a European option

$$\xi_t = \frac{\partial P(t,S_t)}{\partial s} \text{ with } P(t,s) = E^Q[P(T,S_T)|S_t = s]$$

delta-hedge

UNIVERSITEIT GENT

Complete market

Hedging in Black-Scholes model

- $dS_t = \sigma S_t dW_t$ (martingale measure Q, no interest rate)
- perfect replication by self-financing strategies
- martingale representation

$$P(T, S_T) = E^Q[P(T, S_T)] + \int_0^T Z_t dW_t = E^Q[P(T, S_T)] + \int_0^T \xi_t dS_t$$

- where in case of a European option

$$\xi_t = \frac{\partial P(t, S_t)}{\partial s} \text{ with } P(t, s) = E^Q[P(T, S_T) | S_t = s]$$

delta-hedge

UNIVERSITEIT GENT

Complete market

Hedging in Black-Scholes model

- $dS_t = \sigma S_t dW_t$ (martingale measure Q, no interest rate)
- perfect replication by self-financing strategies
- martingale representation

$$P(T, S_T) = E^Q[P(T, S_T)] + \int_0^T Z_t dW_t = E^Q[P(T, S_T)] + \int_0^T \xi_t dS_t$$

- where in case of a European option

$$\xi_t = \frac{\partial P(t, S_t)}{\partial s} \text{ with } P(t, s) = E^Q[P(T, S_T) | S_t = s]$$

delta-hedge

UNIVERSITEIT GENT

Complete market

Hedging in Black-Scholes model

- $dS_t = \sigma S_t dW_t$ (martingale measure Q, no interest rate)
- perfect replication by self-financing strategies
- martingale representation

$$P(T, S_T) = E^Q[P(T, S_T)] + \int_0^T Z_t dW_t = E^Q[P(T, S_T)] + \int_0^T \xi_t dS_t$$

- where in case of a European option

$$\xi_t = \frac{\partial P(t, S_t)}{\partial s} \text{ with } P(t, s) = E^Q[P(T, S_T) | S_t = s]$$

delta-hedge

UNIVERSITEIT GENT

Incomplete market

- jumps, stochastic volatility or trading constraints
- martingale representation above does not hold
- ‘every claim attainable and replicated by self-financing strategy’ is not valid
- relax one of these two conditions
- hedging is an approximation problem

UNIVERSITEIT GENT

Incomplete market

- utility maximization: non-linear pricing/hedging rule

$$\max_{\xi} E\left[U(c + \int_0^T \xi_t dS_t - H)\right]$$

- quadratic hedging: linear pricing/hedging rule

$$\min_{\xi} E\left[(c + \int_0^T \xi_t dS_t - H)^2\right] \quad \text{(mean-variance)}$$

$$\min_{\xi} E\left[(C_T - C_t)^2 \middle| \mathcal{F}_t\right] \quad \text{((local) risk minimization)}$$

optimal hedging portfolio (if exists) is L^2-projection of H onto the (linear) subspace of hedgeable claims

UNIVERSITEIT GENT

Outline

UNIVERSITEIT GENT

Quadratic hedging

- finding optimal hedging portfolio $\Leftrightarrow$ finding GKW-decomposition or finding FS-decomposition
- Martingale case: easy to determine ξ which is same for RM and MVH (η differs)
- Semimartingale case = martingale + drift
  - LRM: general solution
  - MVH: no general solution due to self-financing condition

UNIVERSITEIT GENT

Quadratic hedging

- finding optimal hedging portfolio $\Leftrightarrow$ finding GKW-decomposition or finding FS-decomposition
- Martingale case: easy to determine ξ which is same for RM and MVH (η differs)
- Semimartingale case $=$ martingale $+$ drift
  - LRM: general solution
  - MVH: no general solution due to self-financing condition

UNIVERSITEIT GENT

Risk-minimization

- S: local martingale under measure P
- T-contingent claim $H \in L^2(P)$
- not self-financing strategy but mean self-financing strategy, i.e. cost process is martingale
- H-admissible strategy: value process has terminal value H
- value process V of discounted portfolio:

$$V_t = E[H|\mathcal{F}_t]$$

UNIVERSITEIT GENT

GKW-decomposition

- Föllmer and Sondermann (1986): solution to risk-minimization problem can be found by

Galtchouk-Kunita-Watanabe decomposition

$$H = E[H] + \int_0^T \xi_u dS_u + L_T$$

with L local martingale orthogonal to S

- by martingale property

$$V_t = E[H|\mathcal{F}_t] = E[H] + \int_0^t \xi_u dS_u + L_t$$

- Hedging strategy: $\varphi = (\xi_t, V_t - \xi_t S_t)$

UNIVERSITEIT GENT

GKW-decomposition

- Föllmer and Sondermann (1986): solution to risk-minimization problem can be found by

Galtchouk-Kunita-Watanabe decomposition

$$H = E[H] + \int_0^T \xi_u dS_u + L_T$$

with L local martingale orthogonal to S

- by martingale property

$$V_t = E[H|\mathcal{F}_t] = E[H] + \int_0^t \xi_u dS_u + L_t$$

- Hedging strategy: $\varphi = (\xi_t, V_t - \xi_t S_t)$

UNIVERSITEIT GENT

GKW-decomposition

- Föllmer and Sondermann (1986): solution to risk-minimization problem can be found by

Galtchouk-Kunita-Watanabe decomposition

$$H = E[H] + \int_0^T \xi_u dS_u + L_T$$

with L local martingale orthogonal to S

- by martingale property

$$V_t = E[H|\mathcal{F}_t] = E[H] + \int_0^t \xi_u dS_u + L_t$$

- Hedging strategy: $\varphi = (\xi_t, V_t - \xi_t S_t)$

UNIVERSITEIT GENT

GKW-decomposition

- Föllmer and Sondermann (1986): solution to risk-minimization problem can be found by

Galtchouk-Kunita-Watanabe decomposition

$$H = E[H] + \int_0^T \xi_u dS_u + L_T$$

with L local martingale orthogonal to S

- by martingale property

$$V_t = E[H|\mathcal{F}_t] = E[H] + \int_0^t \xi_u dS_u + L_t$$

- Hedging strategy: $\varphi = (\xi_t, V_t - \xi_t S_t)$

UNIVERSITEIT GENT

GKW-decomposition

Orthogonal

X is orthogonal to Y ($X \perp Y$) $\Leftrightarrow$ $[X, Y]$ is a local martingale

with $[X, Y] = XY - Y \cdot X - X \cdot Y = XY - \int Y dX - \int X dY$

$\Rightarrow$ compensator of $[X, Y]$: $\langle X, Y \rangle = 0$

Remark: $\langle \cdot, \cdot \rangle$ is measure dependent!

use $\langle \cdot, \cdot \rangle$ to determine ξ from GKW-decomposition

UNIVERSITEIT GENT

GKW-decomposition

Orthogonal

X is orthogonal to Y ($X \perp Y$) $\Leftrightarrow$ $[X, Y]$ is a local martingale

with $[X, Y] = XY - Y \cdot X - X \cdot Y = XY - \int Y dX - \int X dY$

$\Rightarrow$ compensator of $[X, Y]$: $\langle X, Y \rangle = 0$

Remark: $\langle \cdot, \cdot \rangle$ is measure dependent!

use $\langle \cdot, \cdot \rangle$ to determine ξ from GKW-decomposition

UNIVERSITEIT GENT

GKW-decomposition

- computation of ξ from GKW-decomposition:

$$dV_t = \xi_t dS_t + dL_t$$

$$d\langle V,S\rangle_t = \xi_t d\langle S,S\rangle_t + \underbrace{d\langle L,S\rangle_t}_{=0} \quad \Leftrightarrow \quad \xi_t = \frac{d\langle V,S\rangle_t}{d\langle S,S\rangle_t}$$

- risk process and remaining risk:

$$R_t(\varphi) = E[(C_T(\varphi) - C_t(\varphi))^2|\mathcal{F}_t] = E[(L_T - L_t)^2|\mathcal{F}_t]$$

UNIVERSITEIT GENT

GKW-decomposition

- computation of ξ from GKW-decomposition:

$$dV_t = \xi_t dS_t + dL_t$$

$$d\langle V,S\rangle_t = \xi_t d\langle S,S\rangle_t + \underbrace{d\langle L,S\rangle_t}_{=0} \quad \Leftrightarrow \quad \xi_t = \frac{d\langle V,S\rangle_t}{d\langle S,S\rangle_t}$$

- risk process and remaining risk:

$$R_t(\varphi) = E[(C_T(\varphi) - C_t(\varphi))^2|\mathcal{F}_t] = E[(L_T - L_t)^2|\mathcal{F}_t]$$

UNIVERSITEIT GENT

Local risk-minimization

- $S = S_0 + M + B$: one-dimensional P-semimartingale with
  - M: square-integrable local martingale, $M_0 = 0$
  - B: predictable process with finite variation
- Not possible to find a risk-minimizing strategy (Schweizer (1988))

UNIVERSITEIT GENT

Local risk-minimization

- $S = S_0 + M + B$: one-dimensional P-semimartingale with
  - M: square-integrable local martingale, $M_0 = 0$
  - B: predictable process with finite variation
- Not possible to find a risk-minimizing strategy (Schweizer (1988))

UNIVERSITEIT GENT

Local risk-minimization

Schweizer (1990, 1991, 2008)

- minimization of the risk $R_t(\varphi) = E[(C_T(\varphi) - C_t(\varphi))^2 | \mathcal{F}_t]$ replaced by

New criterion

$$\liminf_{n\to\infty} r^{\tau_n}(\varphi, \Delta) \geq 0$$

with

$$r^{\tau}[\varphi, \Delta](\omega, t) := \sum_{t_i, t_{i+1} \in \tau} \frac{R_{t_i}(\varphi + \Delta|_{(t_i, t_{i+1}]}) - R_{t_i}(\varphi)}{E[\langle M\rangle_{t_{i+1}} - \langle M\rangle_{t_i} | \mathcal{F}_{t_i}]} \mathbb{1}_{(t_i, t_{i+1}]}(t)$$

- riskiness of cost process measured locally in time

UNIVERSITEIT GENT

Local risk-minimization

- Schweizer (1991, 2008): H-admissible strategy φ is LRM strategy iff φ is mean-self-financing and martingale $C(\varphi)$ is orthogonal to martingale part M of price process S.
- under assumptions
  - (A1) $\langle M \rangle$ is P-a.s. stricly increasing on $[0, T]$
  - (A2) B is continuous
  - (A3) B is absolutely continuous w.r.t. $\langle M \rangle$ with density λ satisfying $E[\langle \int \lambda dM \rangle] < \infty$

  LRM strategy φ follows from FS-decomposition of $H \in L^2(P)$

UNIVERSITEIT GENT

FS-decomposition

Föllmer-Schweizer decomposition

$$H = H_0 + \int_0^T \xi_u^{FS} dS_u + L_T^{FS}$$

with L^{FS} local martingale orthogonal to M

- How? Recall that $C = V - \xi \cdot S$ is P-martingale and P-orthogonal to M
- Answer: define equivalent martingale measure Q such that C is also Q-martingale

Minimal martingale measure Q related to a P-semimartingale

The martingale measure such that any local martingale orthogonal to M under P remains local martingale under Q

UNIVERSITEIT GENT

FS-decomposition

Föllmer-Schweizer decomposition

$$H = H_0 + \int_0^T \xi_u^{\mathrm{FS}} dS_u + L_T^{\mathrm{FS}}$$

with L^{FS} local martingale orthogonal to M

- How? Recall that $C = V - \xi \cdot S$ is P-martingale and P-orthogonal to M
- Answer: define equivalent martingale measure Q such that C is also Q-martingale

***Minimal martingale measure Q* related to a P-semimartingale**

The martingale measure such that any local martingale orthogonal to M under P remains local martingale under Q

FS-decomposition/LRM

LRM strategy φ is given by

$$\varphi_t = (\xi_t^{\text{FS}}, V_t - \xi^{\text{FS}} S_t)$$

where

$$V_t = E^Q[H|\mathcal{F}_t] = E^Q[H] + \int_0^t \xi_u^{\text{FS}} dS_u + L_t^{\text{FS}}$$

UNIVERSITEIT GENT

FS-decomposition/LRM

- computation of ξ^{FS} from FS-decomposition:

$$dV_t^M + dV_t^B = dV_t = \xi_t^{\text{FS}} dS_t + dL_t^{\text{FS}} = \xi_t^{\text{FS}}(dM_t + dB_t) + dL_t^{\text{FS}}$$

$$d\langle V^M, M\rangle_t = \xi_t^{\text{FS}} d\langle M, M\rangle_t + \underbrace{d\langle L^{\text{FS}}, M\rangle_t}_{=0}$$

$$\Leftrightarrow \quad \xi_t^{\text{FS}} = \frac{d\langle V^M, M\rangle_t}{d\langle M, M\rangle_t}$$

UNIVERSITEIT GENT

FS-decomposition/LRM

- computation of ξ^{FS} from FS-decomposition:

$$dV_t^M + dV_t^B = dV_t = \xi_t^{FS} dS_t + dL_t^{FS} = \xi_t^{FS}(dM_t + dB_t) + dL_t^{FS}$$

$$d\langle V^M, M\rangle_t = \xi_t^{FS} d\langle M, M\rangle_t + \underbrace{d\langle L^{FS}, M\rangle_t}_{=0}$$

$$\Leftrightarrow \quad \xi_t^{FS} = \frac{d\langle V^M, M\rangle_t}{d\langle M, M\rangle_t}$$

UNIVERSITEIT GENT

Outline

UNIVERSITEIT GENT

GKW versus FS

- GKW-decomposition

$$E^Q[H|\mathcal{F}_t] = E^Q[H] + (\xi \cdot S)_t + L_t \text{ with } \langle L, S\rangle^Q = 0$$

  L is Q-local martingale

- FS-decomposition

$$E^Q[H|\mathcal{F}_t] = E^Q[H] + (\xi^{FS} \cdot S)_t + L_t^{FS} \text{ with } \langle L^{FS}, M\rangle = 0$$

  L^{FS} is P-local martingale

- L^{FS} is Q-local martingale by definition of MMM.
  Question 1: Is L^{FS} orthogonal to S, i.e. $\langle L^{FS}, S\rangle^Q = 0$?
  Question 2: Is L P-martingale orthogonal to M?

UNIVERSITEIT GENT

GKW versus FS

- GKW-decomposition

$$E^Q[H|\mathcal{F}_t] = E^Q[H] + (\xi \cdot S)_t + L_t \text{ with } \langle L, S\rangle^Q = 0$$

  L is Q-local martingale
- FS-decomposition

$$E^Q[H|\mathcal{F}_t] = E^Q[H] + (\xi^{FS} \cdot S)_t + L_t^{FS} \text{ with } \langle L^{FS}, M\rangle = 0$$

  L^{FS} is P-local martingale
- L^{FS} is Q-local martingale by definition of MMM.
  Question 1: Is L^{FS} orthogonal to S, i.e. $\langle L^{FS}, S\rangle^Q = 0$?
  Question 2: Is L P-martingale orthogonal to M?

UNIVERSITEIT GENT

GKW versus FS

- $L^{FS} = L + (\xi - \xi^{FS}) \cdot S$
- relation between

$$\xi = \frac{d\langle V, S\rangle^Q}{d\langle S, S\rangle^Q} \quad \text{and} \quad \xi^{FS} = \frac{d\langle V^M, M\rangle}{d\langle M, M\rangle}?$$

- from GKW-decomposition: $dV_t = \xi_t dS_t + L_t$
-

$$\xi^{FS} = \xi + \frac{d\langle L, M\rangle}{d\langle M, M\rangle}$$

UNIVERSITEIT GENT

GKW versus FS

- $L^{FS} = L + (\xi - \xi^{FS}) \cdot S$
- relation between

$$\xi = \frac{d\langle V, S\rangle^Q}{d\langle S, S\rangle^Q} \quad \text{and} \quad \xi^{FS} = \frac{d\langle V^M, M\rangle}{d\langle M, M\rangle}?$$

- from GKW-decomposition: $dV_t = \xi_t dS_t + L_t$

$$d\langle V^M, M\rangle = \xi d\langle M, M\rangle + \langle L, M\rangle$$

-

$$\xi^{FS} = \xi + \frac{d\langle L, M\rangle}{d\langle M, M\rangle}$$

UNIVERSITEIT GENT

GKW versus FS

- $L^{FS} = L + (\xi - \xi^{FS}) \cdot S$
- relation between

$$\xi = \frac{d\langle V, S\rangle^Q}{d\langle S, S\rangle^Q} \quad \text{and} \quad \xi^{FS} = \frac{d\langle V^M, M\rangle}{d\langle M, M\rangle}?$$

- from GKW-decomposition: $dV_t = \xi_t dS_t + L_t$

$$\frac{d\langle V^M, M\rangle}{d\langle M, M\rangle} = \xi \frac{d\langle M, M\rangle}{d\langle M, M\rangle} + \frac{d\langle L, M\rangle}{d\langle M, M\rangle}$$

-

$$\xi^{FS} = \xi + \frac{d\langle L, M\rangle}{d\langle M, M\rangle}$$

UNIVERSITEIT GENT

GKW versus FS

- $L^{FS} = L + (\xi - \xi^{FS}) \cdot S$
- relation between

$$\xi = \frac{d\langle V, S\rangle^Q}{d\langle S, S\rangle^Q} \quad \text{and} \quad \xi^{FS} = \frac{d\langle V^M, M\rangle}{d\langle M, M\rangle}?$$

- from GKW-decomposition: $dV_t = \xi_t dS_t + L_t$

$$\frac{d\langle V^M, M\rangle}{d\langle M, M\rangle} = \xi \frac{d\langle M, M\rangle}{d\langle M, M\rangle} + \frac{d\langle L, M\rangle}{d\langle M, M\rangle}$$

- $$\xi^{FS} = \xi + \frac{d\langle L, M\rangle}{d\langle M, M\rangle}$$

UNIVERSITEIT GENT

GKW versus FS

- $\langle L^{FS}, S\rangle^Q = \underbrace{\langle L, S\rangle^Q}_{=0} + (\xi - \xi^{FS}) \cdot \langle S, S\rangle^Q = 0 \Leftrightarrow \xi = \xi^{FS}$
- $\xi^{FS} = \xi \Leftrightarrow \langle L, M\rangle = 0$
  $\Leftrightarrow L$ is P-martingale orthogonal to M
- Question 1 $\Leftrightarrow$ Question 2 $\Leftrightarrow$ GKW and FS coincide under MMM

UNIVERSITEIT GENT

GKW versus FS

- $\langle L^{\mathrm{FS}}, S\rangle^Q = \underbrace{\langle L, S\rangle^Q}_{=0} + (\xi - \xi^{\mathrm{FS}}) \cdot \langle S, S\rangle^Q = 0 \Leftrightarrow \xi = \xi^{\mathrm{FS}}$
- $\xi^{\mathrm{FS}} = \xi \Leftrightarrow \langle L, M\rangle = 0$
  $\Leftrightarrow$ L is P-martingale orthogonal to M
- Question 1 $\Leftrightarrow$ Question 2 $\Leftrightarrow$ GKW and FS coincide under MMM

GKW versus FS

- $\langle L^{\mathrm{FS}}, S\rangle^Q = \underbrace{\langle L, S\rangle^Q}_{=0} + (\xi - \xi^{\mathrm{FS}}) \cdot \langle S, S\rangle^Q = 0 \Leftrightarrow \xi = \xi^{\mathrm{FS}}$
- $\xi^{\mathrm{FS}} = \xi \Leftrightarrow \langle L, M\rangle = 0$
  $\Leftrightarrow$ L is P-martingale orthogonal to M
- Question 1 $\Leftrightarrow$ Question 2 $\Leftrightarrow$ GKW and FS coincide under MMM

UNIVERSITEIT GENT

Continuous case

- S is continuous process $\Rightarrow$ GKW and FS coincide under MMM
  Föllmer & Schweizer (1991) proved preservation of orthogonality
  answer to question 1: $\langle L^{\text{FS}}, S\rangle^{Q} = 0$
- M is also continuous

$$\Rightarrow \langle L, M\rangle = [L, M] = [L, S] = \langle L, S\rangle^{Q} = 0$$
$$\Leftrightarrow \xi^{\text{FS}} = \xi \Leftrightarrow L^{\text{FS}} = L$$

$$\boxed{\text{GKW} = \text{FS}}$$

UNIVERSITEIT GENT

Continuous case

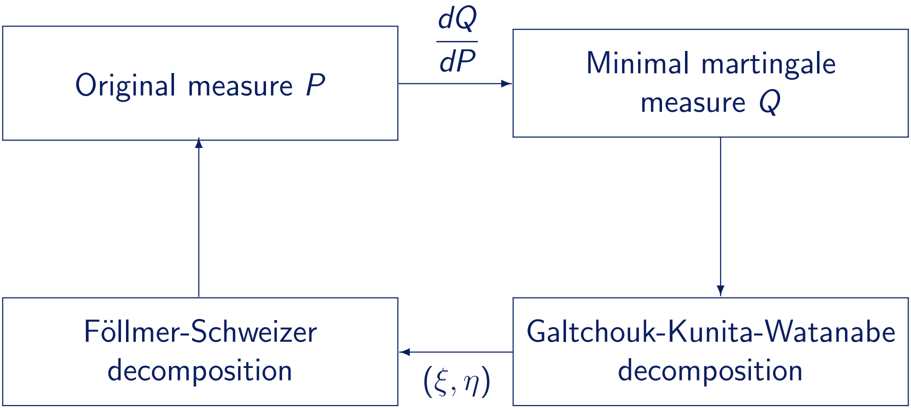
- S is continuous process $\Rightarrow$ GKW and FS coincide under MMM
  Föllmer & Schweizer (1991) proved preservation of orthogonality
  answer to question 1: $\langle L^{\mathrm{FS}}, S\rangle^Q = 0$
- M is also continuous

$$\Rightarrow \langle L, M\rangle = [L, M] = [L, S] = \langle L, S\rangle^Q = 0$$
$$\Leftrightarrow \xi^{\mathrm{FS}} = \xi \Leftrightarrow L^{\mathrm{FS}} = L$$

$$\boxed{\mathrm{GKW} = \mathrm{FS}}$$

UNIVERSITEIT GENT

Continuous case



UNIVERSITEIT GENT

Discontinuous case

- S is discontinuous process $\Rightarrow$ GKW and FS DO NOT coincide under MMM
- orthogonality not preserved from P to Q or vice versa

$$\langle L, S \rangle^Q = 0 \not\Rightarrow \langle L, M \rangle = 0$$

Is Q-local martingale L a P-local martingale?

- question of orthogonality by definition fomulated as question of being martingale:

$$\langle L, M \rangle = 0 \Leftrightarrow [L, M] \text{ is } P\text{-local martingale}$$

$$\boxed{\text{GKW} \neq \text{FS}}$$

UNIVERSITEIT GENT

Discontinuous case

- S is discontinuous process $\Rightarrow$ GKW and FS DO NOT coincide under MMM
- orthogonality not preserved from P to Q or vice versa

$$\langle L, S\rangle^Q = 0 \not\Rightarrow \langle L, M\rangle = 0$$

Is Q-local martingale L a P-local martingale?

- question of orthogonality by definition fomulated as question of being martingale:

$$\langle L, M\rangle = 0 \Leftrightarrow [L, M] \text{ is } P\text{-local martingale}$$

$$\boxed{\text{GKW} \neq \text{FS}}$$

UNIVERSITEIT GENT

Discontinuous case

- S is discontinuous process $\Rightarrow$ GKW and FS DO NOT coincide under MMM
- orthogonality not preserved from P to Q or vice versa

$$\langle L, S\rangle^{Q} = 0 \not\Rightarrow \langle L, M\rangle = 0$$

Is Q-local martingale L a P-local martingale?

- question of orthogonality by definition fomulated as question of being martingale:

$$\langle L, M\rangle = 0 \Leftrightarrow [L, M] \text{ is } P\text{-local martingale}$$

$$\boxed{\text{GKW} \neq \text{FS}}$$

UNIVERSITEIT GENT

Preservation

in terms of predictable characteristics an additional condition different from orthogonality condition on Q-local martingale to be a P-local martingale is required

Main proposition (Choulli, Vandaele, V)

L Q-local martingale, then L is P-local martingale if and only if

$$\lambda'_t c_t \beta_t + \int \left[\lambda'_t x - \lambda'_t \Delta\langle M\rangle_t \lambda_t\right] W_t(x) F_t(dx) = 0.$$

Note: Use uniqueness of the representation theorem

UNIVERSITEIT GENT

Relationship

ξ^{FS} in terms of ξ (Choulli, Vandaele, V)

$(\widetilde{\beta}, \widetilde{f}, \widetilde{g}, L^{\perp})$: quadruplet associated with L under Q, then

$$\xi^{\mathrm{FS}} - \xi = \widetilde{\Phi} \qquad L^{\mathrm{FS}} = L - \widetilde{\Phi} \cdot S,$$

with

$$\widetilde{\Phi} := \Sigma^{\mathrm{inv}} \int x\widetilde{f}(x)[\lambda' x - \lambda' \Delta\langle M\rangle\lambda] F(dx),$$

and Σ^{inv} is the Moore-Penrose pseudoinverse of Σ

$$\Sigma := c + \int xx' F(dx)$$

UNIVERSITEIT GENT

FS-decomposition

Predictable characteristics (Choulli, Vandaele, V)

Consider a square-integrable $\mathcal{F}_T$-measurable random variable H, and denote by $\left(H_0, \xi^{\mathrm{FS}}, L^{\mathrm{FS}}\right)$ its FS-decomposition components. Then the following holds

$$\xi^{\mathrm{FS}} = \Sigma^{inv} \left\{ c\widetilde{\phi} + \int x\widetilde{f}(x)F(dx) \right\} \quad \text{and} \quad L^{\mathrm{FS}} = V - \xi^{\mathrm{FS}} \cdot S.$$

Here $(\widetilde{\phi}, \widetilde{f}, \widetilde{g}, \widetilde{K}^{\perp})$: quadruplet associated with V^M, and Σ is a random symmetric matrix given by

$$\Sigma := c + \int xx' F(dx).$$

Outline

UNIVERSITEIT GENT

Practical example

- $S = S_0 + S^c + x \star (\mu - \nu) + B$ with μ a random measure and ν its P-compensator, $dB_t = bdt$
- H is contingent claim
- $V_t = E^Q[H \mid \mathcal{F}_t] = f(t, S_t)$ with f a $C^{1,2}$-function
- Itô-formula

$$V_t = V_0 + \int_0^t f_x(s, S_{s-})dS + \int_0^t [f_t(s, S) + \frac{1}{2} f_{xx}(s, S_{s-})]ds + \sum_{0<s\leq t} [f(s, S_{s-}) - f(s, S_{s-}) - f_x(s, S_{s-})\Delta S_s]$$

UNIVERSITEIT GENT

Practical example

- P-martingale part of V:

$$V^M = f_x(\cdot, S_-) \cdot S^c + [f(\cdot, S_- + x) - f(\cdot, S_-)] \star (\mu - \nu)$$

- FS-decomposition of H

$$\xi^{\mathrm{FS}} = \frac{\left[c f_x(\cdot, S_-) + \int_{\mathbb{R}} x[f(\cdot, S_- + x) - f(\cdot, S_-)]F(dx)\right]}{c + \int_{\mathbb{R}} x^2 F(dx)}$$

$$L^{\mathrm{FS}} = V - V_0 - \xi^{\mathrm{FS}} \cdot S$$

UNIVERSITEIT GENT

Counterexample

- one-dimensional discounted process modelled as Lévy process:

$$S_t := S_0\mathcal{E}(\overline{S})_t, \quad \overline{S}_t := \sigma W_t + \gamma\tilde{p}_t + \mu t$$

  - p: standard Poisson process with intensity 1
  - $\tilde{p}_t = p_t - t$, $0 \leq t \leq T$: compensated Poisson process
  - W: standard Brownian motion
  - $S_0 > 0$, $\sigma > 0$, $\gamma > -1$, $0 \neq \mu\gamma < \sigma^2 + \gamma^2$
- decomposition of S: $S = S_0 + M + B$?

$$dS_t = S_{t-}d\overline{S}_t$$

UNIVERSITEIT GENT

Counterexample

- decomposition of S: $S = S_0 + M + B$?

$$dM_t = S_{t-}(\sigma dW_t + \gamma d\tilde{p}_t), \quad dB_t = \mu S_{t-} dt$$

- minimal martinagle measure Q? density given by

$$Z := \mathcal{E}(\underbrace{-\lambda \cdot M}_{N}) \text{ with } dB = \lambda d\langle M \rangle$$

- for this model

$$\lambda_t = \frac{1}{S_{t-}} \frac{\mu}{\sigma^2 + \gamma^2}, \quad N_t = \sigma_1 W_t + \gamma_1 \tilde{p}_t, \; \sigma_1 := \frac{-\mu\sigma}{\sigma^2 + \gamma^2}, \; \gamma_1 := \frac{-\mu\gamma}{\sigma^2 + \gamma^2}$$

UNIVERSITEIT GENT

Counterexample

- decomposition of S: $S = S_0 + M + B$?

$$dM_t = S_{t-}(\sigma dW_t + \gamma d\tilde{p}_t), \quad dB_t = \mu S_{t-} dt$$

- minimal martingagle measure Q? density given by

$$Z := \mathcal{E}(\underbrace{-\lambda \cdot M}_{N}) \text{ with } dB = \lambda d\langle M\rangle$$

- for this model

$$\lambda_t = \frac{1}{S_{t-}} \frac{\mu}{\sigma^2 + \gamma^2}, \quad N_t = \sigma_1 W_t + \gamma_1 \tilde{p}_t, \ \sigma_1 := \frac{-\mu\sigma}{\sigma^2 + \gamma^2}, \ \gamma_1 := \frac{-\mu\gamma}{\sigma^2 + \gamma^2}$$

UNIVERSITEIT GENT

Counterexample

- European put option with payoff: $H = (K - S_T)_+$
- by independent increments of S

$$V_t = f(t, S_t) \text{ with } f(t,x) = E^Q\left[(K - x\frac{S_T}{S_t})_+\right]$$

- distribution function of S?

$$S_t = S_0\mathcal{E}(\overline{S})_t = S_0 e^{\overline{S}_t - \overline{S}_0 - \frac{1}{2}\langle \overline{S}^c\rangle_t} \prod_{s\leq t}(1+\Delta\overline{S}_s)e^{-\Delta\overline{S}_s}$$

with $\langle \overline{S}^c\rangle_t = \langle \sigma W\rangle_t = \sigma^2 t$
$\Delta\overline{S}_s = \gamma\Delta\tilde{p}_s = \gamma\Delta p_s$ being zero or one

- $$S_t = S_0 e^{\sigma W_t + \tilde{p}_t \log(1+\gamma) + (\mu - \frac{1}{2}\sigma^2 + \log(1+\gamma) - \gamma)t}$$

UNIVERSITEIT GENT

Counterexample

- strictly increasing distribution function in y, $y = \log x$

$$F(s,y) = Q(\frac{S_s}{S_0} \leq x) = Q(\log(S_s) - \log(S_0) \leq y)$$
$$= Q(\sigma W_S + \log(1+\gamma)\tilde{p}_s + \overline{\mu} s \leq y)$$

- by stationarity property of S, for $x > 0$:

$$f(t,x) = E^Q\left[(K - x\frac{S_T}{S_t})_+\right] = xE^Q\left[(\frac{K}{x} - \mathcal{E}(\overline{S})_{T-t})_+\right]$$
$$= x\int_{-\infty}^{\log\frac{K}{x}} (\frac{K}{x} - e^y) dF(T-t,y)$$
$$= KF(T-t, \log\frac{K}{x}) - x\int_{-\infty}^{\log\frac{K}{x}} e^y F_y(T-t,y)dy$$

UNIVERSITEIT GENT

Counterexample

- $f \in C^{1,2}((0,T) \times (0,+\infty))$, apply Itô to $f(t, S_t)$ and V is Q-martingale

$$V_t = V_0 + \int_0^t f_x(u, S_{u-}) dS_u + (\Gamma \cdot \tilde{p}^Q)_t$$
$$\tilde{p}^Q := p_t - (1+\gamma_1)t$$
$$\Gamma_u := f(u, S_{u-}(1+\gamma)) - f(u, S_{u-}) - f_x(u, S_u)\gamma S_{u-}$$

- ξ from GKW-decomposition

$$\xi = \frac{d\langle V, S\rangle^Q}{d\langle S, S\rangle^Q} = f_x + \Gamma \frac{d\langle \tilde{p}^Q, S\rangle^Q}{d\langle S, S\rangle^Q}$$

UNIVERSITEIT GENT

Counterexample

GKW-decomposition: $V = V_0 + \xi \cdot S + L$

- ξ given by

$$\xi_t = f_x(t, S_{t-}) + \Gamma_t \frac{d\langle \tilde{p}^Q, S\rangle_t^Q}{d\langle S, S\rangle_t^Q}$$
$$= f_x(t, S_{t-}) + \frac{\Gamma_t}{S_{t-}} \frac{\gamma(1+\gamma_1)}{\sigma^2 + \gamma^2(1+\gamma_1)}$$

- $V_t = V_0 + \int_0^t f_x(u, S_{u-}) dS_u + (\Gamma \cdot \tilde{p}^Q)_t$

$$L = \Gamma \cdot \tilde{p}^Q - \frac{\Gamma}{S_-} \frac{\gamma(1+\gamma_1)}{\sigma^2 + \gamma^2(1+\gamma_1)} \cdot S$$

UNIVERSITEIT GENT

Counterexample

Result

The FS-decomposition of H and the GKW-decomposition under Q of V differ.

Proof

- difference between ξ^{FS} and ξ:

$$\xi^{\text{FS}} - \xi = \frac{d\langle L, M\rangle}{d\langle M, M\rangle} = \frac{\Gamma}{S_-} \frac{\mu\gamma^2\sigma^2}{(\sigma^2+\gamma^2)^2(\sigma^2+\gamma^2(1+\gamma_1))}$$

- compute $\Gamma_t = f(t, S_{t-}(1+\gamma)) - f(t, S_{t-}) - f_x(t, S_t)\gamma S_{t-}$

$$f_x(t,x) = -\int_{\infty}^{\log \frac{K}{x}} e^y F_y(T-t, y)dy$$

Counterexample

Proof (continued)

- plug in expressions for f and f_x:

$$\Gamma_t = f(t, S_{t-}(1+\gamma)) - f(t, S_{t-}) - f_x(t, S_t)\gamma S_{t-}$$
$$= \int_{s_1(t)}^{s_2(t)} [K - S_{t-}(1+\gamma)e^y] F_y(T-t, y) dy$$

with $s_1(t) := \log \frac{K}{S_{t-}}$ and $s_2(t) := s_1(t) - \log(1+\gamma)$

- $\Gamma \neq 0$ for $\gamma \neq 0$:

$(-1 <)\gamma < 0$: $s_1 < s_2$ and $[K - S_{t-}(1+\gamma)e^y] F_y(T-t, y) > 0$

$\gamma > 0$: $s_1 > s_2$ and

$$\Gamma_t = \int_{s_2(t)}^{s_1(t)} [-K + S_{t-}(1+\gamma)e^y] F_y(T-t, y) dy > 0$$

UNIVERSITEIT GENT

Counterexample

Proof (continued)

- plug in expressions for f and f_x:

$$\Gamma_t = f(t, S_{t-}(1+\gamma)) - f(t, S_{t-}) - f_x(t, S_t)\gamma S_{t-}$$
$$= \int_{s_1(t)}^{s_2(t)} [K - S_{t-}(1+\gamma)e^y] F_y(T-t, y) dy$$

with $s_1(t) := \log \frac{K}{S_{t-}}$ and $s_2(t) := s_1(t) - \log(1+\gamma)$

- $\Gamma \neq 0$ for $\gamma \neq 0$:

$(-1 <)\gamma < 0$: $s_1 < s_2$ and $[K - S_{t-}(1+\gamma)e^y]F_y(T-t, y) > 0$

$\gamma > 0$: $s_1 > s_2$ and

$$\Gamma_t = \int_{s_2(t)}^{s_1(t)} [-K + S_{t-}(1+\gamma)e^y] F_y(T-t, y) dy > 0$$

UNIVERSITEIT GENT

Counterexample

Proof (continued)

- plug in expressions for f and f_x:

$$\Gamma_t = f(t, S_{t-}(1+\gamma)) - f(t, S_{t-}) - f_x(t, S_t)\gamma S_{t-}$$
$$= \int_{s_1(t)}^{s_2(t)} [K - S_{t-}(1+\gamma)e^y] F_y(T-t, y) dy$$

with $s_1(t) := \log \frac{K}{S_{t-}}$ and $s_2(t) := s_1(t) - \log(1+\gamma)$

- $\Gamma \neq 0$ for $\gamma \neq 0$:

$(-1 <)\gamma < 0$: $s_1 < s_2$ and $[K - S_{t-}(1+\gamma)e^y]F_y(T-t, y) > 0$

$\gamma > 0$: $s_1 > s_2$ and

$$\Gamma_t = \int_{s_2(t)}^{s_1(t)} [-K + S_{t-}(1+\gamma)e^y] F_y(T-t, y) dy > 0$$

□

Outline

UNIVERSITEIT GENT

T. Choulli, N. Vandaele, and M. Vanmaele.
The Föllmer-Schweizer decomposition: Comparison and description.
Stochastic Processes and their Applications, 120(6):853–872, 2010.

H. Föllmer, M. Schweizer.
Hedging of contingent claims under incomplete information.
Applied Stochastic Analysis, 5:389–414, 1991.

H. Föllmer, D. Sondermann.
Hegding of non-redundant contingent claims.
Contributions to Mathematical Economics, 205–223, 1986.

M. Schweizer.
Hedging of options in a general semimartingale model.
PhD thesis, Swiss Federal Institution of Technology Zürich, 1988.

M. Schweizer.
Risk-minimality and orthogonality of martingales.
Stochastics and Stochastic Reports, 30(1):123-131, 1990.

M. Schweizer.
Option hedging for semimartingales.
Stochastic Processes and their Applications, 37:339-363, 1991.

M. Schweizer.
Local risk minimization for multidimensional assets and payment streams.
Banach Center Publications, 83:213-229, 20081.

Thank you for your attention

This study was supported by a grant of Research Foundation-Flanders