Goodness-of-Fit Testing with Empirical Copulas

Sami Umut Can
John Einmahl
Roger Laeven

Department of Econometrics and Operations Research
Tilburg University

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Overview of Copulas

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- More precisely, a function $C : [0,1]^2 \to [0,1]$ is called a bivariate copula if
  - $C(x,0) = C(0,y) = 0$ for any $x, y \in [0,1]$
  - $C(x,1) = x$, $C(1,y) = y$ for any $x, y \in [0,1]$
  - $C(x_2,y_2) - C(x_1,y_2) - C(x_2,y_1) + C(x_1,y_1) \geq 0$ for any $x_1, x_2, y_1, y_2 \in [0,1]$ with $x_1 \leq x_2$ and $y_1 \leq y_2$

- Sklar's Theorem:
  Let H be a bivariate cdf with continuous marginal cdf's

$$H(x, \infty) = F(x), \quad H(\infty, y) = G(y).$$

  Then there exists a unique copula C such that

$$H(x, y) = C(F(x), G(y)). \tag{1}$$

  Conversely, for any univariate cdf's F and G and any copula C, (1) defines a bivariate cdf H with marginals F and G.

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- C captures the dependence structure of two random variables. It is used for dependence modeling in finance and actuarial science.

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- In other words, we would like to perform a hypothesis test about C, with a null hypothesis of the form $C = C_0$ or $C \in \{C_\theta, \theta \in \Theta\}$. For now, we consider the simple hypothesis ($C = C_0$) only.

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- In other words, we would like to perform a hypothesis test about C, with a null hypothesis of the form $C = C_0$ or $C \in \{C_\theta, \theta \in \Theta\}$. For now, we consider the simple hypothesis $(C = C_0)$ only.
- We would like to use the *empirical copula* to construct a goodness-of-fit test.

- Note that we can write

$$C(x, y) = H(F^{-1}(x), G^{-1}(y)), \quad (x, y) \in [0, 1]^2,$$

with $F^{-1}(x) = \inf\{t \in \mathbb{R} : F(t) \geq x\}$, and similarly for G^{-1}.

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- So a natural way of estimating the copula C is using the *empirical copula*

$$C_n(x,y) = H_n(F_n^{-1}(x), G_n^{-1}(y)), \quad (x,y) \in [0,1]^2,$$

with

$$H_n(x,y) = \frac{1}{n}\sum_{i=1}^{n} \mathbf{1}\{X_i \leq x, Y_i \leq y\},$$

$$F_n(x) = \frac{1}{n}\sum_{i=1}^{n} \mathbf{1}\{X_i \leq x\}, \quad G_n(y) = \frac{1}{n}\sum_{i=1}^{n} \mathbf{1}\{Y_i \leq y\}$$

- It is known that the *empirical copula process*

$$D_n(x,y) = \sqrt{n}(C_n(x,y) - C(x,y)), \quad (x,y) \in [0,1]^2$$

converges weakly in $\ell^\infty([0,1]^2)$ to a C-Brownian pillow, under the assumption that

$$C^x(x,y) \text{ is continuous on } \{(x,y) \in [0,1]^2 : 0 < x < 1\},$$
$$C^y(x,y) \text{ is continuous on } \{(x,y) \in [0,1]^2 : 0 < y < 1\}.$$

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- A C-Brownian sheet $W(x, y)$ is a mean zero Gaussian process with covariance function

$$\text{Cov}[W(x, y), W(x', y')] = C(x \wedge x', y \wedge y'), \quad x, x', y, y' \in [0, 1].$$

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- A C-Brownian pillow $D(x,y)$ is a mean zero Gaussian process that is equal in distribution to the C-Brownian sheet W, conditioned on $W(x,y) = 0$ for any $(x,y) \in [0,1]^2 \setminus (0,1)^2$.

- We have

$$D(x,y) = W(x,y) - C^x(x,y)W(x,1) - C^y(x,y)W(1,y) \\ - (C(x,y) - xC^x(x,y) - yC^y(x,y))W(1,1).$$

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- So we know the asymptotic distribution of the empirical copula process

$$D_n(x,y) = \sqrt{n}(C_n(x,y) - C(x,y)),$$

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  and we can take a functional of D_n (such as the sup over $[0,1]^2$ or an appropriate integral) as a test statistic for a goodness-of-fit test.
- Problem: The asymptotic distribution of D_n, and that of the test statistic, depends on C. We would like to have a *distribution-free* goodness-of-fit test.

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- We use E. Khmaladze's "scanning" idea to transform D into a standard two-parameter Wiener process Z defined on $[0,1]^2$. The same transformation applied to D_n will then produce a process Z_n that will, hopefully, converge to Z.
- Assumptions on C: Continuous first-order partial derivatives on $[0,1]^2 \setminus \{(0,0),(0,1),(1,0),(1,1)\}$, continuous second-order partial derivatives on $(0,1)^2$, strictly positive mixed partial C^{xy} on $(0,1)^2$, and more (to be determined).

- Define a grid $\{(x_i, y_j) : 0 \le i, j \le N\}$ on $[0, 1]^2$ such that

$$0 = x_0 < x_1 < \ldots < x_N = 1$$
$$0 = y_0 < y_1 < \ldots < y_N = 1$$

and define filtrations

$$\mathcal{F}_x(x_i) = \sigma\{D(x_h, y_k) : 0 \le h \le i, 0 \le k \le N\}, \quad 0 \le i \le N$$
$$\mathcal{F}_y(y_j) = \sigma\{D(x_h, y_k) : 0 \le h \le N, 0 \le k \le j\}, \quad 0 \le j \le N$$

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- “Scan” the process D with respect to the filtration $\{\mathcal{F}_x\}$:

$$K_1^{(N)}(x_i, y_j) = \sum_{h=0}^{i-1} \Big(D(x_{h+1}, y_j) - D(x_h, y_j) - E[D(x_{h+1}, y_j) - D(x_h, y_j) | \mathcal{F}_x(x_h)] \Big)$$

- We compute

$$K_1^{(N)}(x_i, y_j) = D(x_i, y_j) - \sum_{h=0}^{i-1} D(x_h, y_j) \left(\frac{E[D(x_h, y_j)D(x_{h+1}, y_j)]}{E[D(x_h, y_j)^2]} - 1 \right)$$

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- Making the x-partitioning finer and finer, we obtain

$$K_1(x, y_j) = D(x, y_j) - \int_0^x D(s, y_j)\xi_1(ds, y_j)$$

as a limit in probability, where ξ_1 is an absolutely continuous measure whose density is determined by C and its first- and second-order derivatives.

- Next, we scan K_1 with respect to the filtration $\{\mathcal{F}_y\}$ and take the limit as the y-partition gets finer and finer:

$$K(x,y) = D(x,y) - \int_0^x D(s,y)\xi_1(ds,y) - \int_0^y D(x,t)\xi_2(x,dt) + \int_0^x \int_0^y D(s,t)\xi_1(ds,t)\xi_2(s,dt),$$

where ξ_2 is another absolutely continuous measure whose density is determined by C and its first- and second-order derivatives.

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• We need a technical lemma about the densities ψ_1 and ψ_2 of the measures ξ_1 and ξ_2, namely that

$$\sup_{(x,y)\in[0,1]^2} x(1-x)|\psi_1(x,y)| < \infty,$$
$$\sup_{(x,y)\in[0,1]^2} y(1-y)|\psi_2(x,y)| < \infty.$$

- **Corollary:** The process

$$Z(x,y) = \int_0^x \int_0^y \frac{1}{\sqrt{C^{xy}(s,t)}} dK(s,t), \quad (x,y) \in [0,1]^2$$

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- We have thus transformed the C-Brownian pillow D into a standard two-parameter Wiener process Z, through a two-step transformation:

$$D \mapsto K \mapsto Z.$$

- We apply the same two-step transformation to D_n, i.e. we define

$$K_n(x,y) = D_n(x,y) - \int_0^x D_n(s,y)\xi_1(ds,y) - \int_0^y D_n(x,t)\xi_2(x,dt) + \int_0^x \int_0^y D_n(s,t)\xi_1(ds,t)\xi_2(s,dt),$$

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- Future: Construct actual test statistics and procedures for goodness-of-fit tests. Consider composite null hypotheses of the form $C \in \{C_\theta : \theta \in \Theta\}$ and consider m-dimensional copulas with $m > 2$.

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- Future: Construct actual test statistics and procedures for goodness-of-fit tests. Consider composite null hypotheses of the form $C \in \{C_\theta : \theta \in \Theta\}$ and consider m-dimensional copulas with $m > 2$.
- Thank you for listening!