Multivariate Risk Modeling

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Quantification of Counterparty Risk Via Bessel Bridges

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Agenda

- CVA and wrong way/right way risk
- Dynamical credit index model
- Calibrating credit index model to CDS data
- Joint modelling of asset prices and credit index
- Applications to oil swap and counterparty risk

The problem

Counterparty Risk

This is the exposure of a bank to a counterparty in some contract should the counterparty default *at some specific time* in the future.

Note: Exposure = max(value, 0) [No exposure if we owe them!]

‘Exposure’ can be quantified using various risk-measures:

- Quantile of loss distribution (VAR)
- Expected shortfall (ES)
- Expected Positive Exposure (EPE)

These are all functions of the post-default distribution of contract value.

Counterparty Valuation Adjustment (CVA)

the ‘fair’ amount to be charged for counterparty risk

The problem

Right-way/Wrong-way risk

To quantify the counterparty risk the connection between exposure and the assumed counterparty default at t needs to be taken into account, i.e. the *conditional distribution* of the value of the contract given that the default event occurs at time t.

Right-way risk: Negative correlation between default and exposure.

Wrong-way risk: Positive correlation.

Problem

The problem of quantification/measurement of counterparty risk is split into two steps:

- Develop a dynamical credit risk model for the timing of default of the counterparty that matches exactly given CDS quotes.
- Develop a joint asset price-credit risk model

Dynamical credit index model

- Following the approach of John Hull we model the default time τ_Y of the counterparty by the first passage time below zero

$$\tau_0^Y = \inf\{t \geq 0 : Y_t < 0\}$$

  of a **credit-index process** Y.

- Given a (risk-neutral) default time distribution H and a family $\mathcal{Y}$ of stochastic processes, the model for the credit index process $Y \in \mathcal{Y}$ should be such that

$$P(\tau_0^Y \leq t) = H(t)$$

  for all $t \geq 0$.

A drifted BM model

Given a distribution function H on $\mathbb{R}^+ \cup \{\infty\}$ with density function h, can we find a distribution function F on $\mathbb{R}^+$ such that τ_0^X has distribution H where

$$X_t = A + \nu t + B_t$$

where $A \sim F$ and A is independent of the Brownian motionB_t?

A drifted BM model

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Answer: Necessary condition is that there exists an distribution function F s.t.

$$\boxed{\mathcal{L}H(\psi(\theta)) = \mathcal{L}F(\theta)} \qquad (*)$$

where

$$\psi(\theta) = \frac{1}{2}\theta^2 - \nu\theta$$

The equation $(*)$ does not always admit a solution as the left-hand side may not be a Laplace transform of a probability distribution.

A drifted BM model

Let $a > 0$ be fixed and

$$X_t = a + \nu t + B_t.$$

The *moment generating function* is given by

$$\mathbb{E}_a[e^{-q\tau_0^X}] = e^{-a\,\Phi(q)}, \quad q > 0,$$

where $\Phi(q) = \nu + \sqrt{\nu^2 + 2q}$.
The corresponding distribution function is K_a with density

$$k_a(t) = \frac{1}{\sqrt{2\pi}} a t^{-3/2} e^{-(a-\nu t)^2/2t}.$$

It is also known that:

$$\mathbb{P}_a\left(\tau_0^X < \infty\right) = \exp\left((2\nu a)_-\right),$$

A drifted BM model

We are looking for F that satisfies

$$H(t) = \mathbb{P}[\tau_0^X \le t] = \int_0^\infty \mathbb{P}_a[\tau_0^X \le t] F(da).$$

Taking the Laplace-Stieltjes transform in t gives

$$\mathcal{L}H(q) = \int_0^\infty \mathbb{E}_a[e^{-q\tau_0^X}] F(da) = \int_0^\infty e^{-a\,\Phi(q)} F(da).$$

We know Φ has inverse $q = \psi(\theta) = \frac{1}{2}\theta^2 - \nu\theta$ so if F exists it must satisfy the stated relation $(*)$.

Key example: exponential distribution

Here the left-hand side of $(*)$ is

$$\frac{\lambda}{\psi(\theta)+\lambda} = \frac{2\lambda}{\theta_+ - \theta_-}\left(\frac{1}{\theta-\theta_+} - \frac{1}{\theta-\theta_-}\right).$$

This is the LT of a distribution on $\mathbb{R}^+$ if and only if $\psi(\theta) > -\lambda$ for all $\theta \geq 0$ and $\nu^2 - \lambda \geq 0$. We obtain the following solutions for the density $f_\lambda(x) = (d/dx)F(x)$.

- $\nu < -\sqrt{2\lambda}: \quad f_\lambda(x) = \frac{2\lambda}{\theta_+ - \theta_-}(e^{\theta_+ x} - e^{\theta_- x})$.
- $\nu = -\sqrt{2\lambda}: \quad f_\lambda(x) = 2\lambda x e^{-x\sqrt{2\lambda}}$.

Other examples that are explicitly solvable are

- mixtures of exponentials and
- convolutions of exponentials

A linear Gaussian process

Take a deterministic non-negative function $\sigma(t)$ and a BM $\tilde{B}_t$ and define the process Y_t by

$$dY_t = \nu\sigma^2(t)dt + \sigma(t)d\tilde{B}_t, \quad Y_0 = A. \tag{1}$$

Theorem. Let $\lambda > 0$ and $A \sim f_\lambda$. Let H be a distribution on $\mathbb{R}^+$ with density h and hazard function

$$\gamma(t) = \frac{h(t)}{\int_t^\infty h(s)ds} = \frac{h(t)}{\overline{H}(t)}.$$

Define Y_t by (1) with

$$\sigma(t) = \sqrt{\frac{\gamma(t)}{\lambda}}.$$

Then $\tau_0^Y \sim H$ where $\quad \tau_0^Y = \inf\{t : Y_t \leq 0\}.$

A linear Gaussian process

The process Y_t is equal in law to $X(I_t)$ where $X(t) = A + \nu t + B_t$ as above and

$$I_t = \int_0^t \sigma^2(s)ds.$$

Since $h/\overline{H} = -(d/dt)\log \overline{H}$ we have

$$I_t = -\frac{1}{\lambda}\log \overline{H}(t).$$

Since $Y_t = X(I_t)$ and τ_0^X has exponential distribution, we see that

$$\mathbb{P}[\tau_0^Y > t] = \mathbb{P}[\tau_0^X > I_t] = e^{-\lambda I_t} = e^{\log \overline{H}(t)} = \overline{H}(t).$$

A linear Gaussian process

Assume again

$$dY_t = \nu\sigma^2(t)dt + \sigma(t)d\tilde{B}_t, \quad Y_0 = A. \tag{2}$$

but now let A be a fixed positive constant. In that case there also exists a solution:

Theorem. Let $\nu < 0$ and $A = a > 0$. Let H be a cdf on $\mathbb{R}_+$ with density h, with $H(0) = 0$ and set $s_0 := \inf\{s : H(s) > 0\}$. For $s \geq 0$ define

$$\sigma^2(s) \quad = \quad \begin{cases} 0 & s \leq s_0 \\ \dfrac{h(s)}{k_a\left((K_a^{-1}(H(s))\right)} & s > s_0 \end{cases}$$

Then it holds that $\mathbb{P}_a(\tau_0 \leq t) = H(t), \qquad t \in \mathbb{R}_+.$

A linear Gaussian process

The proof is similar. Note that now

$$\begin{aligned} I_t &= \int_{s_0}^t \sigma^2(s)ds = \int_0^t \frac{1}{k_a\left((K_a^{-1}(H(s))\right)} h(s)\mathrm{d}s \\ &= K_a^{-1}(H(t)) \end{aligned}$$

where k_a denotes the Brownian first passage density to zero starting from a.
Since the processes Y_t and $X(I_t)$ are equal in law and τ_0^X follows distribution K_a under $\mathbb{P}_a$, we see that

$$\mathbb{P}_a[\tau_0^Y \leq t] = \mathbb{P}_a[\tau_0^X \leq I_t] = K_a(K_a^{-1}(H(t))) = H(t).$$

Calibrating Risk-neutral default-time distributions

For CDS contracts written on an underlying name ABC, we assume that premium payments are made at times t_i and the available maturities are $T_j = t_{k(j)}, j = 1, \ldots, n$. For contract j there is an upfront premium π_j^0 and a running premium rate π_j^1 (with accrual factors δ_i). The recovery rate is $R \in (0, 1)$. The 'fair premium' (π_j^0, π_j^1) then satisfies

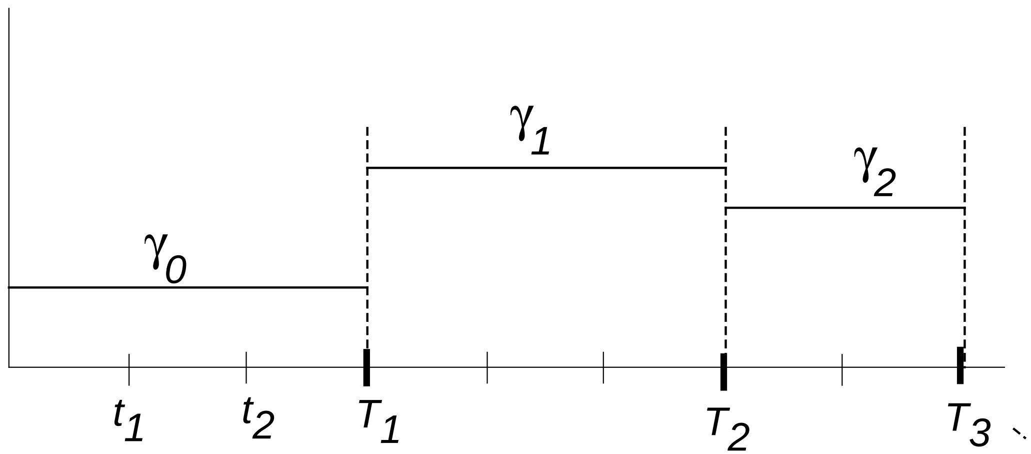
$$\pi_j^0 + \pi_j^1 \sum_{i=0}^{k(j)-1} \delta_i p(0, t_i) \overline{H}(t_i) = (1 - R) \sum_{i=1}^{k(j)} p(0, t_i)(\overline{H}(t_{i-1}) - \overline{H}(t_i)).$$

We take the default distribution to have piecewise-constant hazard rate, i.e.

$$\overline{H}(t) = \exp\left(-\int_0^t \gamma(s) ds\right)$$

where $\gamma(s) = \gamma_i$ for $T_i \leq s < T_{i+1}$ (with $T_0 = 0$.)

Calibrating Risk-neutral default-time distributions



We then back out $\gamma_0, \gamma_1, \ldots$ given the market data $(\pi_1^0, \pi_1^1), (\pi_2^0, \pi_2^1), \ldots$

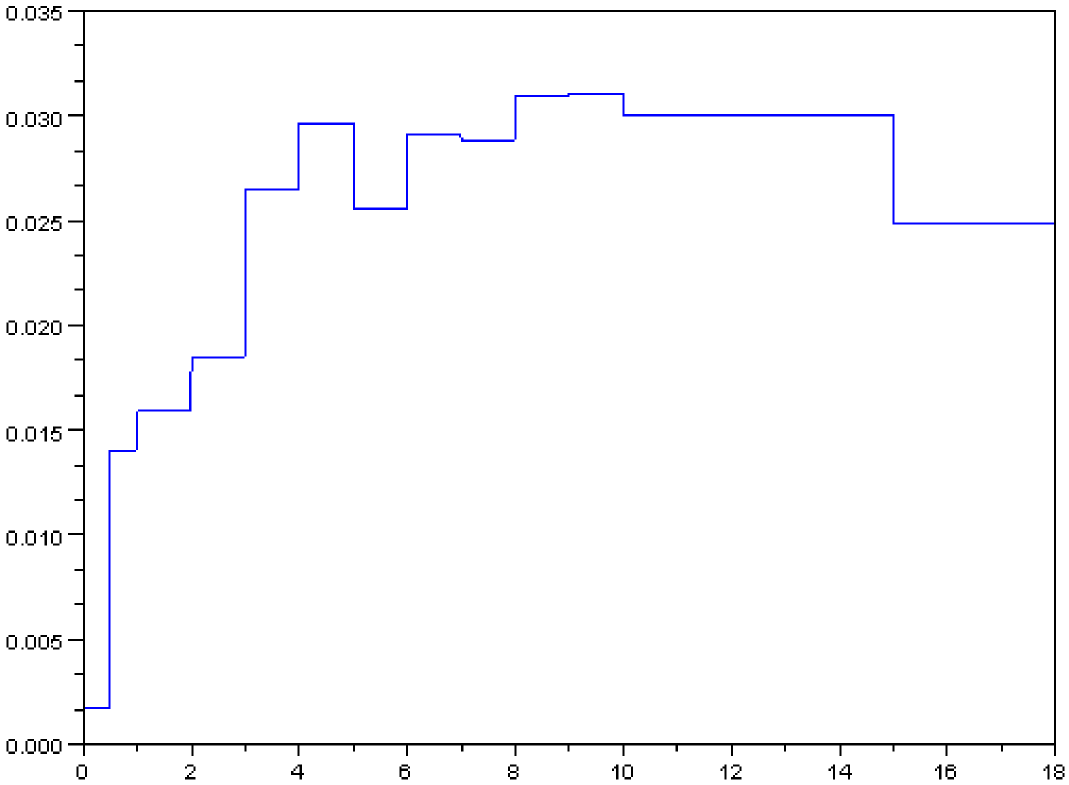
The model Y that probability of hitting 0 exactly given by H is then given by

$$dY_t = \nu\sigma^2(t)dt + \sigma(t)d\tilde{B}_t, \quad Y_0 = A,$$

where $\sigma^2(t) = \gamma(t)/\lambda$, i.e. Y_t has piecewise-constant coefficients.

Calibrating Risk-neutral default-time distributions

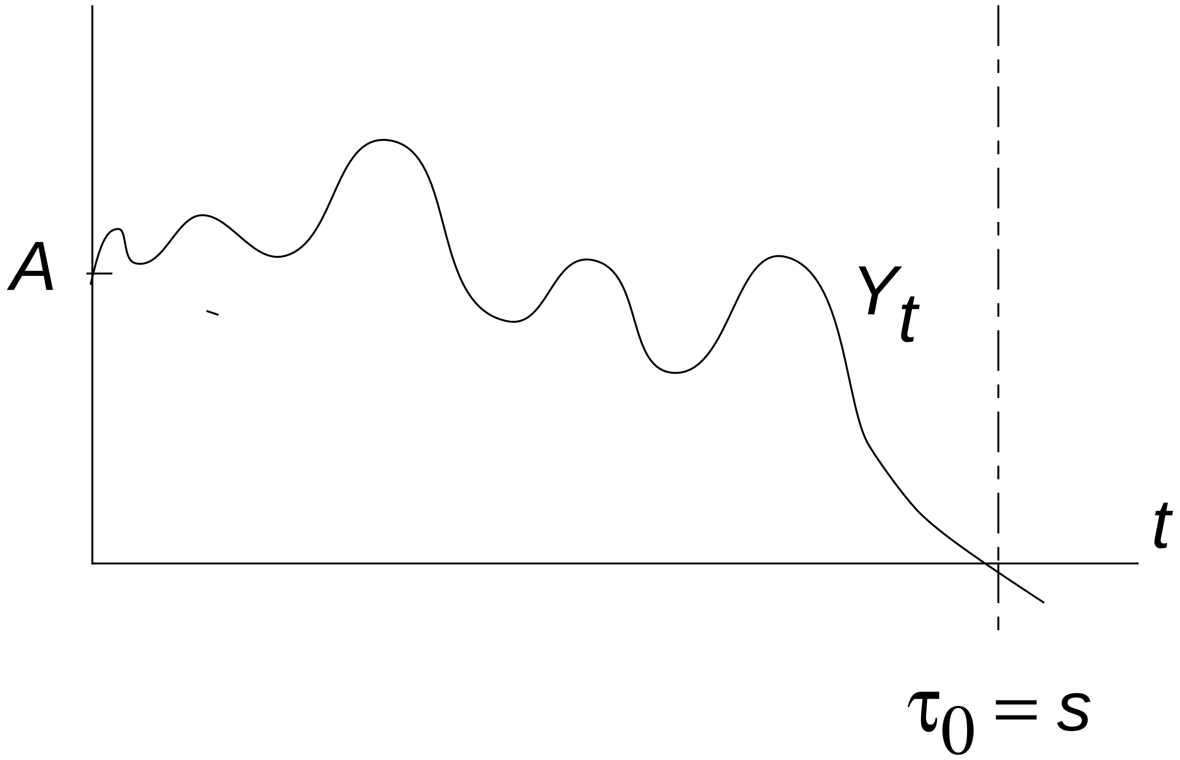
Example: Southwest Airlines.



We have 12 CDS quote for maturities ranging from 6m to 16y.

Models conditioned on default time

To evaluate counterparty risk, we condition on default at a specific time $s > 0$. For path-dependent contracts we need the conditional law of the default risk process Y_t conditioned on the event $(\tau_0^Y = s)$.



Models conditioned on default time: Bessel bridge

The law of $X_t = a + \nu t + B_t$ conditioned to hit 0 for the first time at $\tau_0^X = s$ is equal to that of the 3-dimensional Bessel Bridge from $a \to 0$ on $[0, s]$. We apply the Doob h-transform with h given by

$$h(t, x) \quad = \quad \mathbb{P}_x[\tau_0^X \in [s - dt, s]]/dt = k_x(s - t).$$

This means applying a change of measure $d\mathbb{Q}/d\mathbb{P}|_{\mathcal{F}_t} = h(t, X_t)/h(0, a)$. By Girsanov, the change of drift is (with $h' = \partial h/\partial x$)

$$\frac{h'}{h} = (\log h)' = \left(\log x - \frac{1}{2(s-t)}(x + \nu(s-t))^2\right)' = \frac{1}{x} - \frac{x}{s-t} - \nu.$$

Thus under $\mathbb{Q}$, X_t satisfies the SDE

$$dX_t = \left(\frac{1}{X_t} - \frac{X_t}{s-t}\right) dt + dB_t, \quad t \in [0, s) \qquad (3)$$

Models conditioned on default time: Bessel bridge

The Brownian bridge Z_t from $Z_0 = a$ to $Z_1 = 0$ satisfies

$$dZ_t = -\frac{Z_t}{1-t}dt + dW_t, \quad Z_0 = a.$$

It can also be represented as

$$Z_t = \frac{s-t}{s}a + B_t - \frac{t}{s}B_s$$

where B_t is ordinary Brownian motion.

A result of Bertoin and Pitman states that

$$X =|_{\mathcal{L}} \quad \sqrt{(a(s-t) + X_{1,t})^2 + X_{2,t}^2 + X_{3,t}^2}$$

where $X_i,\ i = 1,2,3$ are independent $0 \to 0$ Brownian Bridges.

This provides us with an efficient simulation method.

Models conditioned on default: general case

Recall that τ_0^Y is the first hitting time of 0 by the process

$$dY_t = \nu\sigma^2(t)dt + \sigma(t)dB_t, \quad Y_0 = A.$$

Proposition. Conditioned on $\tau_0^Y = s > 0$, the process Y_t satisfies

$$\begin{aligned} dY_t &= \left(\frac{1}{Y_t} - \frac{Y_t}{\int_t^s \sigma^2(u)du}\right)\sigma^2(t)dt + \sigma(t)d\tilde{B}_t, \quad t \in (0,s) \\ Y_0 &= A, \end{aligned}$$

where $\tilde{B}$ is Brownian motion and $A \sim F$ is independent of $\tilde{B}$.

The $(\cdots)$ term can alternatively be expressed as

$$\left(\frac{1}{Y_t} - \frac{\lambda Y_t}{\log(\overline{H}(t)/\overline{H}(s))}\right).$$

Case study: Oil swaps and counterparty risk

Oil swaps in the Schwartz & Smith model
In the Schwartz & Smith model the evolution of the risk-adjusted oil spot price S_t is modelled by the SDE

$$\frac{\mathrm{d}S_t}{S_t} = (r_t - \delta_t)\mathrm{d}t + \sigma \mathrm{d}W_t, \tag{4}$$

where r_t is the risk-free interest rate, σ a volatility parameter, W_t a Wiener process, and δ_t is a stochastic convenience yield given by

$$\mathrm{d}\delta_t = (\theta(t) - \mu\delta_t)\mathrm{d}t + \beta \mathrm{d}Z_t, \tag{5}$$

where $\theta(\cdot)$ is the time-dependent reversion level, $\mu > 0$ is the rate of mean reversion, β a volatility parameter, and Z a Wiener process that is correlated with W, with $\langle W, Z\rangle_t = \rho_{WZ}t$.

Case study: Oil swaps and counterparty risk

The value $G(t,T)$ at time $t \leq T$ of a contract for the delivery of one unit of oil at T is equal to

$$G(t,T) = \mathbb{E}\left[\mathrm{e}^{-\int_t^T r(s)\mathsf{d}s} S_T \middle| \mathcal{F}_t\right],$$

where $\mathcal{F}_t$ denote the standard filtration generated by $(S_s, s \leq t)$ and $(r_s, s \leq t)$.

As a consequence of the 'affine' nature of the Schwarz-Smith model,

$$G(t,T) = S_t \exp(A(t,T) - \delta_t B_\mu(t,T)) =: G_{t,T}(S_t, \delta_t),$$

where

$$B_\mu(u,t) = \frac{1}{\mu}(1 - e^{-\mu(t-u)}).$$

Case study: Oil swaps and counterparty risk

In an *oil swap* the parties exchange a fixed-rate payment $\delta_i K$ for a floating payment $\delta_i S_{T_i}$ at coupon dates $T_1, , \ldots, T_n$ where δ_i is the accrual factor and S_{T_i} is the oil spot price at T_i. The (payer's) swap value at any time t is

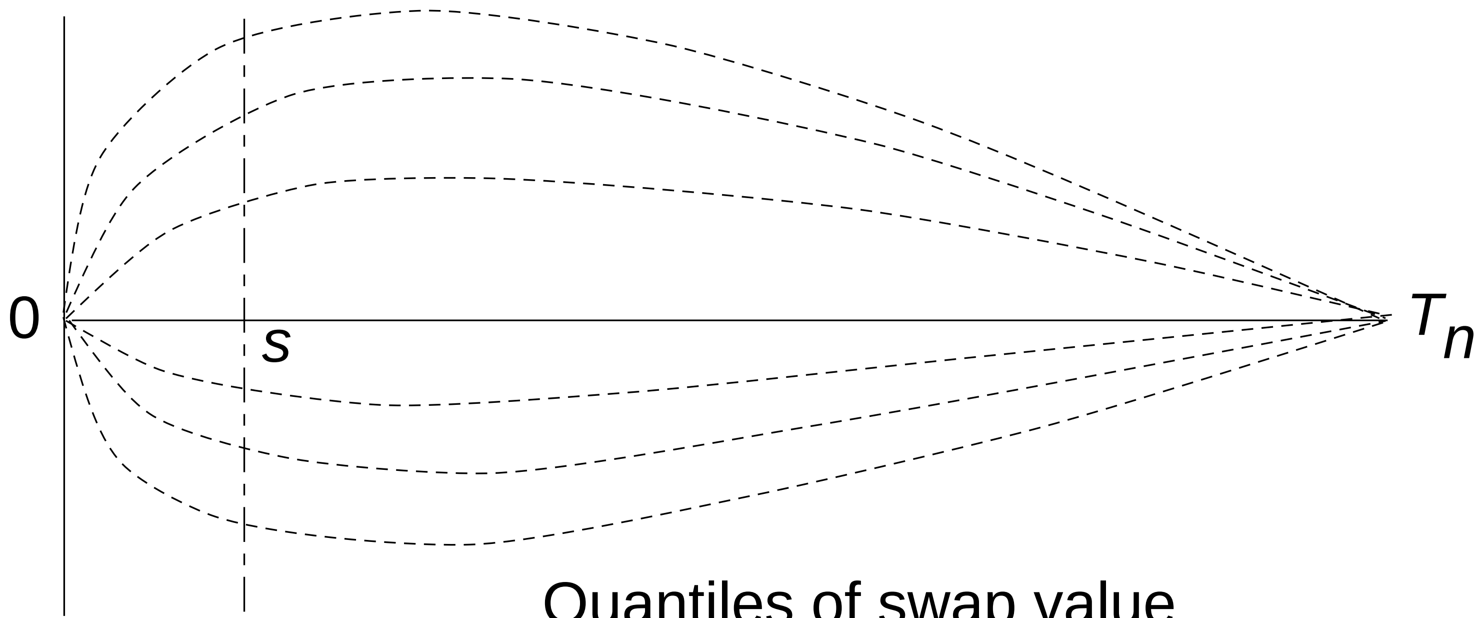
$$V_t = \sum_{i \geq k(t)} \delta_i G(t, T_i) - K \sum_{i \geq k(t)} \delta_i p(t, T_i)$$

where $p(t, T_i)$ is the price at time $t < T_i$ of a zero-coupon risk-free bond with maturity T_i (i.e. a contract that pays $\pounds 1$ at time T_i) and $k(t)$ is the next coupon date after t.

Case study: Oil swaps and counterparty risk

The *swap rate* R_t is the value of K such that $V_t = 0$, i.e.

$$R_t = \frac{\sum_{i \geq k(t)} G(t, T_i)}{\sum_{i \geq k(t)} p(t, T_i)} = \frac{\sum_{i \geq k(t)} G_{t,T_i}(S_t, \delta_t)}{\sum_{i \geq k(t)} p(t, T_i)}. \qquad (6)$$



Quantiles of swap value

Counterparty risk problem: Calculate swap value distribution at $t > 0$ conditional on counterparty default at t.

Case study: Oil swaps and counterparty risk

The essential problem is to get the distribution of $(S(s), \delta(s))$ given that default happens at s. Recall

$$\begin{aligned} \mathrm{d}S_t &= (r_t - \delta_t)S_t\mathrm{d}t + \sigma S_t\mathrm{d}W_t, &(7)\\ \mathrm{d}\delta_t &= (\theta(t) - \mu\delta_t)\mathrm{d}t + \beta\mathrm{d}Z_t. &(8)\end{aligned}$$

We take the counterparty risk model developed above, i.e. the default time is τ_0^Y where Y is the process

$$dY_t = \nu\sigma^2(t)dt + \sigma(t)dB_t, \quad Y_0 = A. \qquad (9)$$

Here Z, W, B are Brownian motions with correlations ρ_{BW}, ρ_{BZ}, and ρ_{BZ} and ρ_{WZ}.

Calibration: $\theta(\cdot)$ is specified such that the forward curve is matched exactly, $(\mu, \sigma, \beta, \rho_{WZ})$ are calibrated from the ATM oil futures option quotes, and $\sigma(\cdot)$ from the counterparty CDS quotes.

Case study: Oil swaps and counterparty risk

The solution of (7)-(8) is

$$\log S_s/S_0 = \tilde{\Xi}_1(s) + [\sigma\rho_{BW} + \beta\rho_{BZ}]\,\Omega_1(s) \qquad (10)$$

$$\delta_s = \tilde{\Xi}_2(s) + \beta\rho_{BZ}\,\Omega_2(s). \qquad (11)$$

Here, fox fixed s, $(\tilde{\Xi}_1(s), \tilde{\Xi}_2(s)) \sim \mathcal{N}(\alpha_s, \Sigma_s^2)$, independent of Y, for some vector α_s and covariance matrix Σ_s^2, and

$$\boxed{(\Omega_1(s), \Omega_2(s)) = \left(\int_0^s \frac{B_\mu(u,s)}{\sigma(u)}\mathrm{d}Y_u, \int_0^s \frac{e^{\mu(u-s)}}{\sigma(u)}\mathrm{d}Y_u\right)} \qquad (12)$$

If we *condition on default at time* s then Y_t satisfies the SDE

$$\boxed{dY_t = \left(\frac{1}{Y_t} - \frac{Y_t}{\int_t^s \sigma^2(u)du}\right)\sigma^2(t)dt + \sigma(t)d\tilde{B}_t, \quad t \in (0, s)} \qquad (13)$$

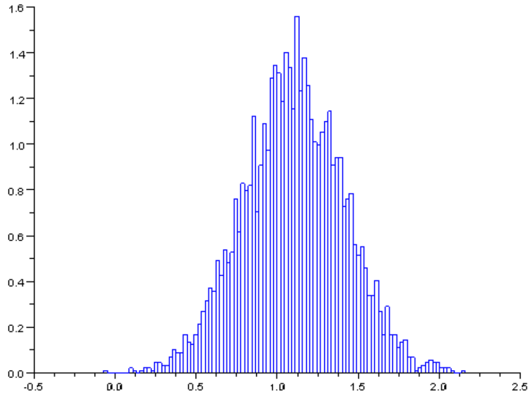
Case study: Oil swaps and counterparty risk

- By Monte Carlo simulation of (10) — (13) we can obtain the empirical distribution of $(S(s), \delta(s))$ at the assumed default time s.

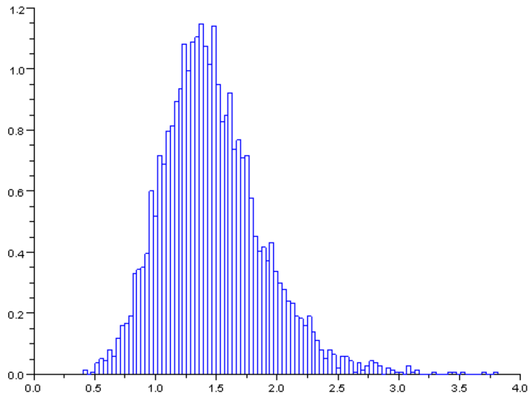
- Since the oil swap rate V_s at time s is a deterministic function of $(S(s), \delta(s))$, we can hence obtain the distribution of the value of the swap conditional on default at time s.

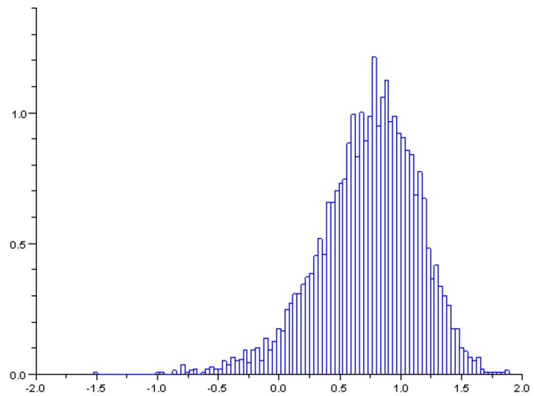
Case study: Oil swaps and counterparty risk

Example: $s = 2.5$ years in a 5-year oil swap with counterparty Southwest Airlines, with $\rho_{BW} = 0$ and different values of ρ_{BZ}.
$\rho_{BZ} = 0$:



$\rho_{BZ} = -0.6, +0.6$





Conclusion

We developed a joint model for asset values and counterparty default risk.

The present approach could be extended and developed in various directions:

- More efficient computational methods need to be developed.
- Multi-asset problems
- Inclusion of credit assets (CDOs,...)
- A consistent procedure would be needed for calibrating these correlation parameters.

Paper

The paper can be downloaded at

`http://ssrn.com/abstract=1722604`