SCALE-FREE PERCOLATION

Maria Deijfen
Stockholm University

Joint work with Remco van der Hofstad and Gerard Hooghiemstra.

EMPIRICAL NETWORKS

- Social relations
- Communication nets
- Internet/www etc.

Propertis:

- Heavy-tailed degree distribution

  Often: Fraction of vertices with degree $> k$ decays like $k^{-\gamma}$ (power-law).
- Short distances between vertices.

INHOMOGENEOUS RANDOM GRAPHS

n vertices

Assign weight $W \sim F$ indep. to each vertex.

$P(\text{edge}\,(i,j)) = f(W_i, W_j)$ $\left[\text{for instance } f(W_i, W_j) = \frac{W_i W_j}{n} \wedge 1\right]$

Edges indep. given the weights.

- F power-law $\Rightarrow$ Limiting degree distribution power-law with same exponent
- Phase-transition for distances:

  $\text{Var}(\text{degree}) < \infty \Rightarrow \log n$

  $\text{Var}(\text{degree}) = \infty \Rightarrow \log\log n$

LONG-RANGE PERCOLATION

$x, y \in \mathbb{Z}^d$

$P(\text{edge } (x,y)) \sim \lambda |x-y|^{\alpha}$ as $|x-y| \to \infty$.

Edges independent.

$d=1$ $\alpha \leq 2 \Rightarrow$ Percolation possible.

$\alpha > 2 \Rightarrow$ No percolation

$d \geq 2$ Focus on how long-range edges affect properties of ∞ component (e.g. distances).

INHOMOGENEOUS L-R PERCOLATION

Assign weight $W_x \geq 0$ to $x \in \mathbb{Z}^d$.

$\{W_x\}$ iid

$$P(\text{edge } (x,y)) = 1 - e^{-\frac{\lambda W_x W_y}{|x-y|^\alpha}} \sim \frac{\lambda W_x W_y}{|x-y|^\alpha}$$

Edges independent given the weights.

α : Long-range parameter.

λ = Percolation parameter.

THE DEGREES

slowly varying

Assume $P(W > w) = w^{-\beta} \cdot L(w)$

D_x = degree of x

$\gamma = \frac{\alpha}{d} \cdot \beta$

slowly varying

$$\boxed{\text{THEOREM. If } \alpha > d \text{ and } \gamma > 1, \text{ then } P(D_0 > s) = s^{-\gamma} \cdot \ell(w)}$$

Note:

$\gamma > 1 \Rightarrow E[D] < \infty$

$\gamma > 2 \Rightarrow \operatorname{Var} D < \infty$

Hence:

- $E[D] < \infty$
- $\operatorname{Var} W < \infty \Rightarrow \operatorname{Var} D < \infty$
- $\operatorname{Var} W = \infty$ but $\operatorname{Var} D < \infty$ possible.

THEOREM. If:

a) $\alpha \leq d$, or

b) $\alpha > d$ and $\gamma \leq 1$,

then $P(D_0 = \infty \mid W_0 > 0) = 1$.

$\therefore$ Infinite degrees for $\alpha \leq d$ or $\gamma \leq 1$.

PERCOLATION

$C(x)$ = component of x

$\theta(\lambda) = P(|C(0)| = \infty)$ percolation probab.

$\lambda_c = \inf\{\lambda : \theta(\lambda) > 0\}$

1. When is $\underbrace{\lambda_c < \infty}_{\text{percolation for large } \lambda}$?

2. When is $\underbrace{\lambda_c > 0}_{\text{no percolation for small } \lambda}$?

Assume: $\alpha > d$ and $\gamma > 1$.

1) <u>When is $\lambda_c < \infty$?</u>

- In $d \geq 2$: As soon as $P(W=0) \neq 1$.
- In $d=1$.

  $\alpha \in (1,2]$: If $P(W \geq \omega) = 1$ some $\omega > 0$.

  $\alpha > 2$: Never, if $\gamma > 2$ (i.e. $\mathrm{Var}\, D < \infty$).

<u>OPEN PROBLEMS</u>

- Find weaker condition in $d=1$, $\alpha \in (1,2]$.
- Can there be percolation in $d=1$ when $\alpha > 2$ and $\gamma \leq 2$?

2) When is $\lambda_c > 0$?

- In $d \geq 2$: $\lambda_c \begin{cases} = 0 & \text{if } \gamma < 2 \\ > 0 & \text{if } \gamma > 2 \end{cases}$

$\therefore$ $\boxed{\lambda_c > 0 \iff \mathrm{Var}\, D < \infty}$ $(\gamma \neq 2)$

- In $d = 1$: $\lambda_c > 0$ if $\gamma > 2$.

DISTANCES

$d(0,x)$ = graph distance between 0 and x

How does $d(0,x)$ grow with $|x|$?

$\gamma = 1$	$\gamma = 2$
$\mathrm{Var}\, D = \infty$	$\mathrm{Var}\, D < \infty$
$d(0,x) \sim \log\log|x|$	$d(0,x) \geq \log|x|$
	(If $\alpha > 2d$: $d(0,x) \geq |x|^{\varepsilon}$)

OPEN PROBLEMS

- Is the log sharp for $\gamma > 2$, $\alpha \in (d, 2d]$?
  When $W \equiv w$: no $\left((\log|x|)^{\Delta},\ \Delta > 1 \right)$
- Determine exponent μ s.th.
  $d(0,x) \sim |x|^{\mu}$ for $\gamma > 2$, $\alpha > 2d$.

SUMMARY

1. Power-law degrees.
   Tail exponent: γ.

2. In $d \geq 2$: $\exists$ non-trivial critical value if $\gamma > 2$, but not if $\gamma < 2$ (then: always percolation).

3. Phase-transition for distances at $\gamma = 2$.