Poisson Hail on a Hot Ground

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We consider a queue where the server is the Euclidean space, and the customers are random closed sets (RACS) of the Euclidean space.

These RACS arrive according to a Poisson rain and each of them has a random service time (in the case of hail falling on the Euclidean plane, this is the height of the hailstone, whereas the RACS is its footprint).

The Euclidean space serves customers at speed 1. The service discipline is a hard exclusion rule: no two intersecting RACS can be served simultaneously and service is in the First In First Out order: only the hailstones in contact with the ground melt at speed 1, whereas the other ones are queued; a tagged RACS waits until all RACS arrived before it and intersecting it have fully melted before starting its own melting.

We prove that it is stable for a sufficiently small arrival intensity, provided the typical diameter of the RACS and the typical service time have finite exponential moments. We also discuss the percolation properties of the stationary regime of the RACS in the queue.

Outline

- Poisson Hail
- Main Result
- Why Is This of Interest
- Idea of the Proof
- Growth Model
- Service and Arrivals
- Bernoulli Hail

Poisson Hail

Arrival:

-Poisson rain on the d dimensional Euclidean space $\mathbb{R}^d$ with intensity λ
(Poisson point process of intensity λ in $\mathbb{R}^{d+1}$)
- Each Poisson arrival, say at location x and time t, brings a customer with two main characteristics:

- A grain C, which is a RACS of $\mathbb{R}^d$ *centered* at the origin. If the RACS is a ball with random radius, its center is that of the ball.
- A random service time σ.

The *mark* (C, σ) of point (x, t) has some given distribution and is independent of everything else.

Service:

-The customer arriving at time t and location x with mark (C, σ) creates a hailstone, with footprint $x + C$ in $\mathbb{R}^d$ and with height σ.

-These hailstones do not move: they are to be melted/served by the Euclidean plane at the location where they arrive in the FCFS order, respecting some hard exclusion rules:
if the footprints of two hailstones have a non empty intersection, then the one arriving second has to wait for the end of the melting/service of the first to start its melting/service.

Once the service of a customer is started, it proceeds uninterrupted at speed 1. Once a customer is served/hailstone fully melted, it leaves the Euclidean space.

Main Result

Let ξ be the (random) diameter of the typical RACS. Assume that the system starts at time $t = 0$ from the empty state, and denote by W_t^x the time to empty the system of all RACS that contain point x and that arrive by time t.

Theorem Assume that the distributions of the random variables ξ^d and σ are *light-tailed*, i.e. there is a positive constant c such that $\mathbf{E}e^{c\xi^d}$ and $\mathbf{E}e^{c\sigma}$ are finite.

Then there exists a positive constant λ_0 (which depends on d and on the joint distribution of ξ and σ) such that, for any $\lambda < \lambda_0$, the model is globally stable.

This means that, for any finite set A in $\mathbb{R}^d$, as $t \to \infty$, the distribution of the random field $(W_t^x, \ x \in A)$ converges weakly to the stationary one.

Comments!

Why Is This of Interest

- Natural generalization, and General mathematical curiosity

- Uniform Bound

Idea of the Proof

It is based on the monotonicity properties.

First, we consider a pure growth model and show that if the input rate λ is below some critical *percolation* level λ_c, then the growth rate is linear.

Second, for the original system. we apply a certain generalization of the *Saturation Rule* to show that if the input rate is so small that the growth rate is less than 1, then the model is stable.

Dynamic Equations:

The *height* H_t^x at location $x \in \mathbb{R}^d$ of the heap made of all RACS arrived before time t (i.e. in the $(0,t)$ interval): for all $t > u \geq 0$,

(1) $$H_t^x = H_u^x + \int_{[u,t)} \left(\sigma_v^x + \sup_{y \in C_v^x} H_v^y - H_v^x \right) N^x(dv),$$

where N^x denotes the Poisson point process on $\mathbb{R}^+$ of RACS arrivals intersecting location x:

$$N^x([a,b]) = \int_{\mathbb{R}^d \times [a,b]} 1_{C_v \cap \{x\} \neq \emptyset} \Phi(dv),$$

and σ_u^x (resp. C_u^x) the canonical height (resp. RAC) mark process of N^x. That is, if the point process N^x has points T_i^x, and if one denotes by (σ_i^x, C_i^x) the mark of point T_i^x, then σ_u^x (resp. C_u^x) is equal to σ_i^x (resp. C_i^x) on $[T_i^x, T_{i+1}^x)$.

Majorants:

- Discretization of Space
- Discretization of Time
- The Branching Upper-Bounds

One of the steps: Construction of the Independent Version.

For a *Boolean Model* with random radii, let C^x be the clump containing $x \in \mathbb{Z}^d$. For $x \neq y \in \mathbb{Z}^d$, either $C^x = C^y$ or these two (random) sets are disjoint, which shows that these two sets are not independent (in the probabilistic meaning).

Lemma Assume that the Boolean Model has a.s. finite clumps only. There exists an extension of the probability space which carries another i.i.d. family $\{(\widehat{C}^z, \widehat{\sigma}^z)\}_{z\in\mathbb{Z}^d}$ such that

1. The following inclusion holds a.s.:

$$C^x \cup C^y \subseteq C^x \cup \widehat{C}^y,$$

2. The random pairs (C^x, σ^x) and $(\widehat{C}^y, \widehat{\sigma}^y)$ are i.i.d., i.e.

$$\begin{aligned} & \mathbf{P}(C^x = A_1, \sigma^x \in B_1, \widehat{C}^y = A_2, \widehat{\sigma}^y \in B_2) \\ = \quad & \mathbf{P}(C^x = A_1, \sigma^x \in B_1)\mathbf{P}(C^y = A_2, \sigma^y \in B_2), \end{aligned}$$

   for all bounded sets A_1, B_1 and A_2, B_2.

Linear Growth:

Theorem If $\lambda < \lambda_c$, then for all $x \in \mathbb{R}^d$, $H_t^x/t \to \kappa(\lambda) = const$ a.s. So, $\kappa(\lambda)$ is the *growth rate*.

Service and Arrivals:

Take any $a < \lambda_c$.

Theorem If $\lambda < \min(\lambda_c, a/\kappa(a))$, then for all $x \in \mathbb{R}^d$, the weak limit of W_t^x, as $t \to \infty$, exists and is finite a.s.

Loynes scheme

Bernoulli Hail on Hot Grid

The state space is $\mathbb{Z}$. All RACS are pairs of neighbouring points/nodes $\{i, i+1\}$, $i \in \mathbb{Z}$ with service time 1. In other words, such a RACS requires 1 unit of time for and simultaneous service from nodes/servers i and $i+1$.

Within each time slot (of size 1), the number of RACS $\{i, i+1\}$ arriving is a Bernoulli-(p) random variable. All these variables are mutually independent.

If a RACS of type $\{i, i+1\}$ and a RACS of type $\{i+1, i+2\}$ arrive in the same time slot, the FIFO tie is solved at random (with probability $1/2$). The system is empty at time 0, and RACS start to arrive from time slot $(0, 1)$ on.

Exact Evolution Equations for the Growth Model

The variable H_n^i is the height of the *last* RACS (segment) of type i that arrived among the set with time index less than or equal to n (namely with index $1 \leq k \leq n$), in the growth model under consideration. If (i, n) is black, then H_n^i is at the same time the height of the maximal height path starting from node (i, n) in $\mathcal{G}(p)$ and the height of the RACS (i, n) in the growth model. If (i, n) is white and the last arrival of type i before time n is k, then $H_n^i = H_k^i$. This is depicted in Figure 1

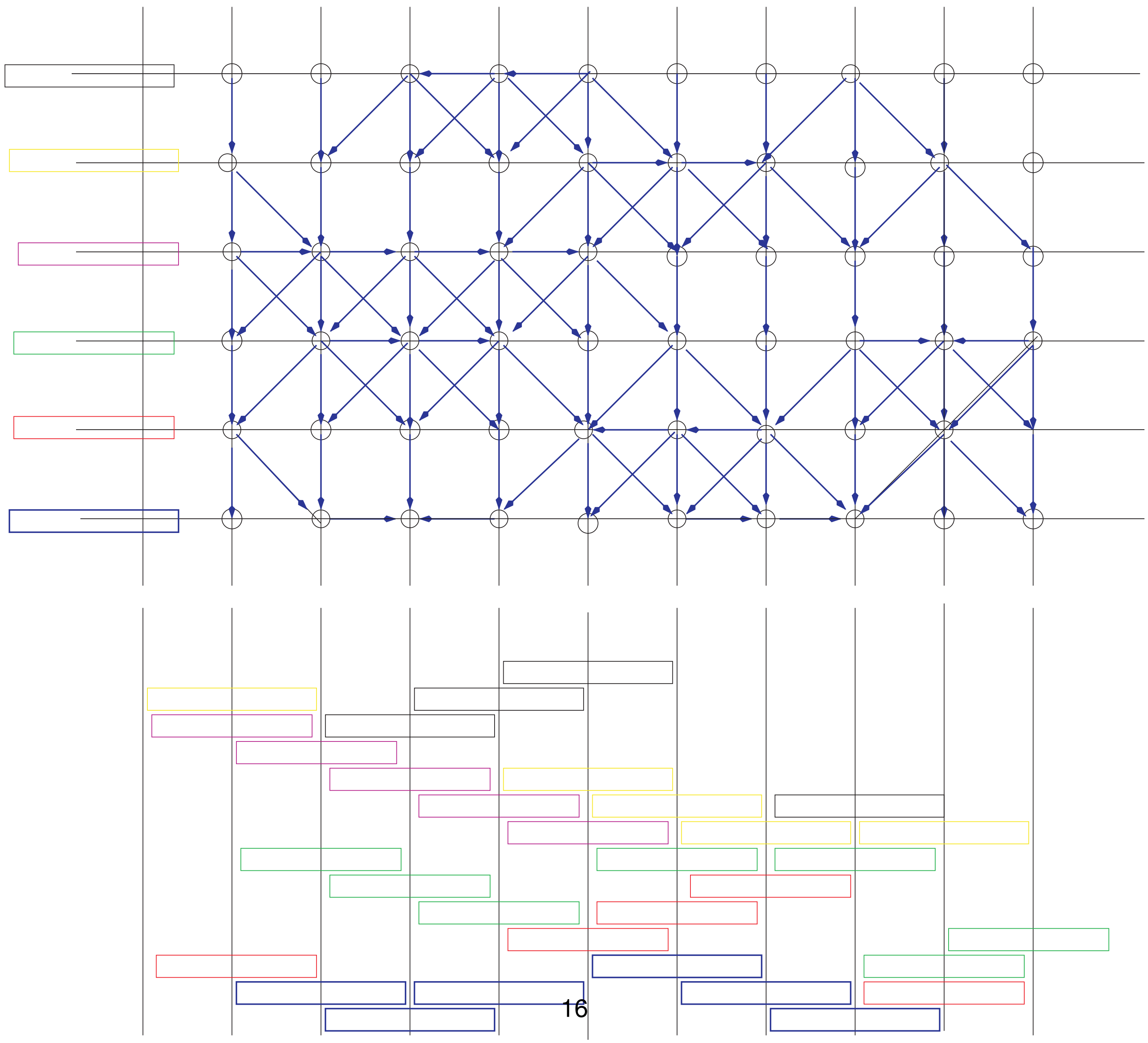

Figure 1: Top: A realization of the random graph $\mathcal{G}(p)$. Only the first 6 time-layers are represented. A black node at (i,n) represents the arrival of a RACS of type i at time n. Bottom: the associated the heap of RACS, with a visualization of the height H_n^i of each RAC.

Open questions

References

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