Asymptotic behavior of the survival probability for a critical branching process in markovian environments

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Abstract

Let $X = (X_n)_{n\geq 0}$ be an irreducible and aperiodic Markov chain on a finite space E with transition matrix P. We denote by G the set of generating functions of probability measures on $\mathbb{N}$, equipped with the topology of simple convergence on $[0,1]$. $\mathcal{B}(G)$ is the Borel σ-algebra on G. Define a Markov chain $(M_n)_{n\geq 0} = (g_n, X_n)_{n\geq 0}$ with values in $G \times E$ and with transition probability Q defined by

$$Q\{(g,\, i), (A \times \{j\})\} = P(i,\, j)\, \overline{F}(i,\, j,\, A), \text{ for } (g,i) \in G \times E,\ A \in \mathcal{B}(G),$$

where $\overline{F}$ is a transition probability from $E \times E$ in the set of probabilities on G. Let $\Omega = (G \times E)^{\mathbb{N}}$ and $\mathcal{F} = \bigotimes^{\mathbb{N}}(\mathcal{B}(G) \bigotimes \mathcal{P}(E))$. We denote by $\mathbb{P}_{(g,i)}$ the unique probability on $(\Omega, \mathcal{F})$, such that for any $(g,i) \in G \times E$, any $n \geq 1$ and any bounded measurable function $f : (G \times E)^n \to \mathbb{R}$, we get

$$\int_\Omega f(M_0(\omega), M_1(\omega), \cdots, M_n(\omega))\mathbb{P}_{(g,i)}(\mathrm{d}\omega)$$
$$= \sum_{(j_1,j_2,\cdots,j_n)\in E^n} P(i,j_1)\cdots P(j_{n-1},j_n) \int_{G^n} f((g,i),(g_1,j_1),\cdots,(g_n,j_n))\overline{F}(i,j_1,\mathrm{d}g_1)\cdots\overline{F}(j_{n-1},j_n,\mathrm{d}g_n).$$

To simplify the notations, $\mathbb{P}_{(Id,i)}$ will be denoted by $\mathbb{P}_i$.
Given $(M_n)_{n\geq 0}$, consider the branching process $(Z_n)_{n\geq 0}$ such that $Z_0 = 1$ and the generating function of Z_n is $g_0 \circ g_1 \circ \cdots \circ g_{n-1}(s)$, $0 \leq s < 1$. The aim of this study is to determine the asymptotic behavior of the survival probability of (Z_n), as $n \to +\infty$.
The branching processes in markovian random environment has been developed by several authors, in particular by K. B. Athreya and S. Karlin [1]. However, the asymptotic behavior of the survival probability of such a process is not yet known.
Consider the function $h : G \to \overline{\mathbb{R}}_+$, $g \mapsto h(g) := g'(1)$. The image of the probability $\overline{F}(i,j,\mathrm{d}x)$ by the map h is denoted by $F(i,j,\mathrm{d}x)$. Assume in this paper the following hypotheses (H):

H1) there exist $\alpha > 0$, such that for all $\lambda \in \mathbb{C}$ satisfying $|\mathrm{Re}\lambda| \leq \alpha$, we have

$$\sup_{(i,j)\in E\times E} |\widehat{F}(i,j,\lambda)| < +\infty, \quad \text{where } \widehat{F}(i,j,\lambda) = \int_{\mathbb{R}} e^{\lambda t} F(i,j,\mathrm{d}t);$$

H2) there exist $n_1 \geq 1$ and $(i_0, j_0) \in E \times E$, such that the measure $\mathbb{P}_{i_0}(X_{n_1} = j, S_{n_1} \in \mathrm{d}x)$ has an absolutely continuous component with respect to the Lebesgue measure on $\mathbb{R}$;

H3) $\sum_{(i,j)\in E\times E} \nu(i)P(i,j) \int_{\mathbb{R}} tF(i,j,\mathrm{d}t) = 0.$

We have

Theorem 1 *Under hypotheses (H), for any $(i,j) \in E \times E$, there exists a constant $\beta_{i,j} > 0$, such that*

$$\lim_{n\to+\infty} \sqrt{n}\, \mathbb{P}_i(Z_n > 0, X_n = j) = \beta_{i,j}.$$

To prove this result, we first prove a local limit theorem for a semi-Markov chain.

References

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