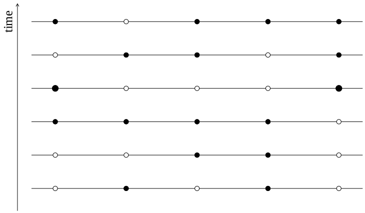
Ancestry in the face of competition: Directed random walk on the directed percolation cluster

Andrej Depperschmidt

work in progress with M. Birkner, J. Černý and N. Gantert

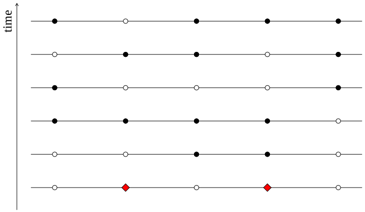
Eindhoven, 15/03/2011

Discrete time contact process on $\mathbb{Z}^d$



- $\{\omega(x,n) : (x,n) \in \mathbb{Z}^d \times \mathbb{Z}\}$ - family of iid $Ber(p)$ rv
- For $m \leq n$ write $(y,m) \to (x,n)$ if there are $(x_m, m), \dots, (x_n, n)$ s. th. $x_m = y$, $x_n = x$, $\|x_k - x_{k-1}\| \leq 1$, $\omega(x_k, k) = 1$.

Discrete time contact process on $\mathbb{Z}^d$

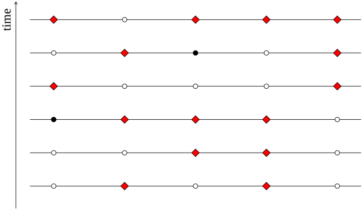


- $\{\omega(x,n) : (x,n) \in \mathbb{Z}^d \times \mathbb{Z}\}$ - family of iid $Ber(p)$ rv
- For $m \le n$ write $(y,m) \to (x,n)$ if there are $(x_m, m), \ldots, (x_n, n)$ s. th. $x_m = y$, $x_n = x$, $\|x_k - x_{k-1}\| \le 1$, $\omega(x_k, k) = 1$.

- Define $(\eta_n)_{n\in\mathbb{Z}}$ by $\eta_n(x) = 1 \iff \exists y \in \mathbb{Z}^d : (y, -\infty) \to (x, n)$
- $p > p_c \;\Rightarrow \mathcal{L}(\eta_0) = \nu$, where ν is the upper invariant measure and

$$\nu\big(\eta_0(x) = 1\big) = \mathbf{P}\big(\exists y \in \mathbb{Z}^d : (y, -\infty) \to (x, 0)\big).$$

Discrete time contact process on $\mathbb{Z}^d$

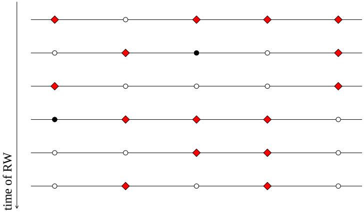


- $\{\omega(x,n) : (x,n) \in \mathbb{Z}^d \times \mathbb{Z}\}$ - family of iid $Ber(p)$ rv
- For $m \le n$ write $(y,m) \to (x,n)$ if there are $(x_m,m),\ldots,(x_n,n)$ s. th. $x_m = y$, $x_n = x$, $\|x_k - x_{k-1}\| \le 1$, $\omega(x_k,k) = 1$.

- Define $(\eta_n)_{n\in\mathbb{Z}}$ by $\eta_n(x) = 1 \iff \exists y \in \mathbb{Z}^d : (y,-\infty) \to (x,n)$
- $p > p_c \;\Rightarrow \mathcal{L}(\eta_0) = \nu$, where ν is the upper invariant measure and

$$\nu\big(\eta_0(x) = 1\big) = \mathbf{P}\big(\exists y \in \mathbb{Z}^d : (y,-\infty) \to (x,0)\big).$$

Random walk on the cluster



- $\mathcal{C} = \{(x,n) \in \mathbb{Z}^d \times \mathbb{Z} : \eta_n(x) = 1\}$
- for $(x,n) \in \mathcal{C}$ define $U(x,n) = \{(y, n-1) \in \mathcal{C} : \|x - y\| \le 1\}$
- On $\mathcal{A}_0 = \{\eta_0(0) = 1\}$ set

$$X_0 = 0, \quad \text{and for } n \ge 1$$

$$\mathbf{P}\big(X_{n+1} = y | X_n = x\big) = \frac{1}{\#U(x,-n)}, \quad \big(y, -(n+1)\big) \in U(x,-n).$$

Strong LLN and annealed CLT

Theorem

We have

$$\mathbf{P}\Big(\frac{1}{n}X_n \to 0 \,\Big|\, \omega\Big) = 1 \quad \textit{for} \quad \mathbf{P}(\,\cdot\,|\mathcal{A}_0)\text{-a.a. } \omega$$

and for all $x \in \mathbb{R}^d$

$$\lim_{n\to\infty} \mathbf{P}\Big(\frac{1}{\sqrt{n}}X_n \le x \,\Big|\, \mathcal{A}_0\Big) = \Phi(x),$$

where Φ *is df of a non-trivial d-dimensional normal distribution.*

Strong LLN and annealed CLT

Theorem

We have

$$\mathbf{P}\Big(\frac{1}{n}X_n \to 0 \,\Big|\, \omega\Big) = 1 \quad \textit{for} \quad \mathbf{P}(\,\cdot\,|\mathcal{A}_0)\text{-a.a. } \omega$$

and for all $x \in \mathbb{R}^d$

$$\lim_{n\to\infty} \mathbf{P}\Big(\frac{1}{\sqrt{n}}X_n \le x \,\Big|\, \mathcal{A}_0\Big) = \Phi(x),$$

where Φ *is df of a non-trivial d-dimensional normal distribution.*

For the proof: Find suitable regeneration structure with finite second moments of the distance between regeneration times.

Previous work

RWRE

- Dolgopyat, Keller and Liverani (2008)
  - Markovian environment, restrictive conditions
- Avena, den Hollander, Redig (2009)
  - LLN, CLT under the cone mixing condition, which does not hold for the contact process (in discrete and in continuous time)

Previous work

RWRE

- Dolgopyat, Keller and Liverani (2008)
  - Markovian environment, restrictive conditions
- Avena, den Hollander, Redig (2009)
  - LLN, CLT under the cone mixing condition, which does not hold for the contact process (in discrete and in continuous time)

Contact process, based on arguments from Kuczek (1989)

- Neuhauser (1992)
  - Ergodic properties of multitype contact process
- Valesin (2010)
  - Coexistence and extinction of one type for multitype contact process

The process of potential ancestors of $(0,0)$

- On $\mathcal{A}_0$ let
$$A_1^{(1)} = (x_1, 1), \ldots, A_1^{(k_1)} = (x_{k_1}, 1)$$
be the randomly ordered set of potential ancestors (PA) of $(0,0)$, i.e.
$$(0,0) \to (x_i, 1), i = 1, \ldots, k_1.$$

- Given $A_n^{(1)}, \ldots, A_n^{(k_n)}$:
  - Let $\widetilde{A}_{n+1}^{(1)}(i), \ldots, \widetilde{A}_{n+1}^{(m_i)}(i)$ be the PA of $A_n^{(i)}$ ordered randomly.
  - Concatenate them to get
$$\widetilde{A}_{n+1}^{(1)}(1), \ldots, \widetilde{A}_{n+1}^{(m_1)}(1), \ldots, \widetilde{A}_{n+1}^{(1)}(k_n), \ldots, \widetilde{A}_{n+1}^{(m_{k_n})}(k_n).$$

  From left to right discard PA appearing for the second, third etc. time. Renumbering the remaining set of PA we get
$$A_{n+1}^{(1)}, \ldots, A_{n+1}^{(k_{n+1})}.$$

Regeneration times

- On $\mathcal{A}_0$ define

$$T_0 := 0, \qquad T_{i+1} := \inf\{n > T_i : (A_n^{(1)}, n) \to \infty\},\ i \geq 0$$
$$\tau_i := T_i - T_{i-1}, \qquad Y_i := X_{T_i} - X_{T_{i-1}},\ i \geq 1.$$

Between T_i and T_{i+1}, (X_n, n) takes the paths from $(A_{T_i}^{(1)}, T_i)$ to $(A_{T_{i+1}}^{(1)}, T_{i+1})$.

Regeneration times

- On $\mathcal{A}_0$ define

$$T_0 := 0, \qquad T_{i+1} := \inf\{n > T_i : (A_n^{(1)}, n) \to \infty\}, \ i \geq 0$$
$$\tau_i := T_i - T_{i-1}, \qquad Y_i := X_{T_i} - X_{T_{i-1}}, \ i \geq 1.$$

Between T_i and T_{i+1}, (X_n, n) takes the paths from $(A_{T_i}^{(1)}, T_i)$ to $(A_{T_{i+1}}^{(1)}, T_{i+1})$.

Proposition

Conditioned on $\mathcal{A}_0$, the sequence $\big((Y_i, \tau_i)\big)_{i\geq 1}$ is iid, $\mathbf{E}[Y_1|\mathcal{A}_0] = 0$ and there are $C, \gamma \in (0, \infty)$ s.th.

$$\mathbf{P}(|Y_1| > n|\mathcal{A}_0), \mathbf{P}(\tau_1 > n|\mathcal{A}_0) \leq Ce^{-\gamma n}.$$

Regeneration times - steps of proof

- Symmetry of Y_1 follows from the construction
- Roughly, independence of $\big((Y_i, \tau_i)\big)_{i \geq 1}$ follows from the fact that the components depend on different time slices of the ω's
- For exponentially decaying tales show (without conditioning on $\mathcal{A}_0$)

$$\tau_1 \leq \sum_{k=1}^{N} \rho_k + \sum_{k=1}^{N-1} \sigma_k, \text{ where}$$

$\rho_k =$ time that a space-time RW needs to find an open site

$\sigma_k =$ depth of cluster started in an open site

N geometrically distributed