Mutually Catalytic Branching Processes and their Relatives[1]

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[1]Keywords: Super Processes, "The" Stepping Stone Model, Generalized Voter Models, Parabolic Anderson Model

Symbiotic Branching SDE

$$\begin{cases} du_t = \sqrt{\gamma u_t v_t} dB_t^1 \\ dv_t = \sqrt{\gamma u_t v_t} dB_t^2 \end{cases}$$

B_t^1, B_t^2 are ϱ-correlated Brownian motions, i.e. $\langle B^1, B^2 \rangle_t = \varrho t$.
The model has two parameters:

- $\gamma > 0$ is called branching rate
- $\varrho \in [-1, 1]$ is called correlation parameter

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$\rightarrow$ solutions interpolate between

- Neutral Wright-Fisher diffusion ($\varrho = -1$)

$$du_t = \sqrt{\gamma u_t (1 - u_t)} dB_t$$

- Linear SDE ($\varrho = 1$)

$$du_t = \gamma u_t dB_t$$

Symbiotic Branching SDE

Theorem (Blath, D., Etheridge '11)

Suppose $\varrho \in (-1, 1)$ *and* $\gamma > 0$*, then*

a) $\lim_{t\to\infty}(u_t, v_t) \overset{a.s.}{=} (u_\infty, v_\infty) \overset{\mathcal{L}}{=} (B^1_\tau, B^2_\tau)$, *where*

$$\tau = \inf\{t \, : \, B^1_t B^2_t = 0\}.$$

"convergence to trivial states"

b)

$$\mathbb{E}[u_t^p] \text{ is bounded in } t \geq 0 \quad \Longleftrightarrow \quad p < p(\varrho)$$

"critical moment curve"

Results do NOT crucially depend on γ, only on ϱ !

Symbiotic Branching SPDE

Take indep. baby symbiotic branching SDEs for each point $k \in \mathbb{Z}^d$
$+$ interaction (smoothing) between neighbors

$$\begin{cases} du_t(k) = \Delta u_t(k)\,dt + \sqrt{\gamma u_t(k) v_t(k)}\,dB^1_t(k) \\ dv_t(k) = \Delta v_t(k)\,dt + \sqrt{\gamma u_t(k) v_t(k)}\,dB^2_t(k) \\ u_0(k) \geq 0 \\ v_0(k) \geq 0 \end{cases}$$

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$\rightarrow$ solutions interpolate between

- "The" neutral stepping stone model ($\varrho = -1$)
- mutually catalytic super processes ($\varrho = 0$)
- parabolic Anderson model with Brownian potential ($\varrho = 1$)
- voter process ($\varrho = -1$ and $\gamma = \infty$)

Symbiotic Branching SPDE

Theorem (spatial version)

$d = 1, 2$

a) *holds in law but NOT almost surely*

b) *holds equally*

$d \geq 3$

a) *does NOT hold (conjecture: only if $\varrho > 0$ and γ large)*

b) *does NOT hold (depends on γ)*

Symbiotic Branching SPDE

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Observation

- *Theorem unifies classical results for boundary cases.*
- *Dependence on d, ϱ and γ.*

How about sending γ to infinity? $\rightarrow$ Mytnik/Klenke for $\varrho = 0$

Symbiotic Branching SPDE with $\gamma = \infty$

Suppose E is the boundary of the first quadrant. Then the $\gamma = \infty$ limiting processes solve

$$\begin{pmatrix} u_t(k) \\ v_t(k) \end{pmatrix} = \begin{pmatrix} u_0(k) \\ v_0(k) \end{pmatrix} + \begin{pmatrix} \int_0^t \Delta u_s(k)\, ds \\ \int_0^t \Delta v_s(k)\, ds \end{pmatrix} + \int_0^t \int_E \left[y_2 \begin{pmatrix} v_{s-}(k) \\ u_{s-}(k) \end{pmatrix} + (y_1 - 1) \begin{pmatrix} u_{s-}(k) \\ v_{s-}(k) \end{pmatrix} \right] (\mathcal{N} - \mathcal{N}')(\{k\}, ds, dy)$$

where $\mathcal{N}$ is a point process on $\mathbb{Z}^d \times \mathbb{R}^+ \times E$ with intensity measure

$$\mathcal{N}'(\{k\}, ds, dy) = I_s(k) ds \nu^{\varrho}(dy),$$

with jump measure ("Lévy measure")

$$\nu^{\varrho}(d(y_1, y_2)) = \begin{cases} C_{\varrho} \dfrac{y_1^{p(\varrho)-1}}{\left(y_1^{p(\varrho)} - 1\right)^2} \, dy_1 & : y_2 = 0 \\ C_{\varrho} \dfrac{y_2^{p(\varrho)-1}}{\left(y_2^{p(\varrho)} + 1\right)^2} \, dy_2 & : y_1 = 0 \end{cases}$$

and time inhomogeneous jump intensity

$$I_s(k) = \begin{cases} \dfrac{\Delta v_{s-}(k)}{u_{s-}(k)} & : u_{s-}(k) > 0 \\ \dfrac{\Delta u_{s-}(k)}{v_{s-}(k)} & : v_{s-}(k) > 0 \end{cases}.$$

$p(\varrho)$ is as in the theorem and $p(\varrho) > 2$ precisely for $\varrho < 0$

Symbiotic Branching SPDE with $\gamma = \infty$ ($\varrho = -1$)

Step 1: Replace ν^{ϱ} by ν^{-1}

$$\begin{pmatrix} u_t(k) \\ v_t(k) \end{pmatrix} = \begin{pmatrix} u_0(k) \\ v_0(k) \end{pmatrix} + \begin{pmatrix} \int_0^t \Delta u_s(k)\, ds \\ \int_0^t \Delta v_s(k)\, ds \end{pmatrix} + \int_0^t \int_E \left[y_2 \begin{pmatrix} v_{s-}(k) \\ u_{s-}(k) \end{pmatrix} + (y_1 - 1) \begin{pmatrix} u_{s-}(k) \\ v_{s-}(k) \end{pmatrix} \right] (\mathcal{N} - \mathcal{N}')(\{k\}, ds, dy)$$

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$$\mathcal{N}'(\{k\}, ds, dy) = I_s(k) ds \nu^{-1}(dy),$$

with jump measure ("Lévy measure")

$$\nu^{-1}(d(y_1, y_2)) = \delta_{(0,1)} + \infty \delta_{(1,0)}$$

and time inhomogeneous jump intensity

$$I_s(k) = \begin{cases} \frac{\Delta v_{s-}(k)}{u_{s-}(k)} & : u_{s-}(k) > 0 \\ \frac{\Delta u_{s-}(k)}{v_{s-}(k)} & : v_{s-}(k) > 0 \end{cases}.$$

Symbiotic Branching SPDE with $\gamma = \infty$ ($\varrho = -1$)

Step 2: skip the infinite atom and cancel the compensation

$$\begin{pmatrix} u_t(k) \\ v_t(k) \end{pmatrix} = \begin{pmatrix} u_0(k) \\ v_0(k) \end{pmatrix} + \int_0^t \int_E y_2 \begin{pmatrix} v_{s-}(k) \\ u_{s-}(k) \end{pmatrix} \mathcal{N}(\{k\}, ds, dy)$$

where $\mathcal{N}$ is a point process on $\mathbb{Z}^d \times \mathbb{R}^+ \times E$ with intensity measure

$$\mathcal{N}'(\{k\}, ds, dy) = I_s(k) ds \nu^{-1}(dy),$$

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$$I_s(k) = \begin{cases} \frac{\Delta v_{s-}(k)}{u_{s-}(k)} & : u_{s-}(k) > 0 \\ \frac{\Delta u_{s-}(k)}{u_{s-}(k)} & : v_{s-}(k) > 0 \end{cases}.$$

Symbiotic Branching SPDE with $\gamma = \infty$ ($\varrho = -1$)

Step 3: assume all initial weights are 1

$$\begin{pmatrix} u_t(k) \\ v_t(k) \end{pmatrix} = \begin{pmatrix} u_0(k) \\ v_0(k) \end{pmatrix} + \int_0^t \int_E y_2 \begin{pmatrix} v_{s-}(k) \\ u_{s-}(k) \end{pmatrix} \mathcal{N}(\{k\}, ds, dy)$$

where $\mathcal{N}$ is a point process on $\mathbb{Z}^d \times \mathbb{R}^+ \times E$ with intensity measure

$$\mathcal{N}'(\{k\}, ds, dy) = I_s(k) ds \nu^{-1}(dy),$$

with jump measure ("Lévy measure")

$$\nu^{-1}(d(y_1, y_2)) = \delta_{(0,1)}$$

and time inhomogeneous jump intensity

$$I_s(k) = \begin{cases} \frac{\Delta v_{s-}(k)}{1} & : u_{s-}(k) > 0 \\ \frac{\Delta u_{s-}(k)}{1} & : v_{s-}(k) > 0 \end{cases}$$
$$= \frac{1}{2d} \text{ number of neighbors of different opinion.}$$

By Itô's formula this is the standard "compound" voter process.

Symbiotic Branching SPDE with $\gamma = \infty$

Results? Not many... only

- coexistence for $\varrho \leq 0$
- rescaling for complete graph migration for $\varrho \leq 0$

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