# Limit Theorems for Voter Model Perturbations

Ted Cox

Syracuse University

YEP VIII, Eurandom, Eindhoven, March 2011

Joint work with **Rick Durrett, Ed Perkins**
(and **Mathieu Merle** if we get that far)

In a nutshell, the goal is to study a class of **interacting particle systems** called **voter model perturbations** via:

- measure-valued limit approach
- hydrodynamic (pde) limit approach

In each case

- the rescaled particle systems converge to something
- the limit can be "inverted" to transfer information back to the particle systems

Plan to give an introduction to the basic tools and methods used.

# Outline of talks

1. spin-flip systems, voter model (graphical representation, duality, martingale problem), super-Brownian motion, convergence

2. voter model perturbations, Lotka-Volterra model, super-Brownian limit, consequences (survival/coexistence)

3. voter model perturbations, hydrodyamic limit, consequences, cooperator/defector model, $d = 2$ Lotka-Volterra model(?)

# Outline Ia

1. Spin-flip systems
   - Basic questions
2. The voter model
   - Graphical construction/duality (first tool)
   - Martingale problem (second tool)
   - Measure-valued point of view
3. References

# Spin-flip systems

Let $\mathbb{Z}^d = d$-dimensional integer lattice.

Consider Feller processes $\boldsymbol{\xi_t}, t \geq 0$ with state space $\{0,1\}^{\mathbb{Z}^d}$,

$$\boxed{\xi_t(x) = \text{type (0 or 1) of "individual" at site } x \in \mathbb{Z}^d \text{ at time } t}$$

Dynamics are determined by a translation invariant **flip rate function** $\boldsymbol{c(x,\xi)} : \mathbb{Z}^d \times \{0,1\}^{\mathbb{Z}^d} \to [0,\infty)$ via

$$P(\xi_{t+h}(x) \neq \xi_t(x) \mid \xi_t) = h\, \boldsymbol{c(x,\xi_t)} + o(h) \text{ as } h \downarrow 0$$

$c(x,\xi)$ is just the rate at which the coordinate at $x$ flips

More formally, ...

$c(x, \xi)$ determines determines a (pre)generator

$$\boldsymbol{G} f(\xi) = \sum_x c(x, \xi)[f(\xi^x) - f(\xi)]$$

where

- $f : \{0, 1\}^{\mathbb{Z}^d} \to \mathbb{R}$ depends on only finitely many coordinates
- $\xi^x$ equals $\xi$ except at $x$, where $\xi^x(x) = 1 - \xi(x)$

Liggett (1972) gives conditions on rate functions $c(x, \xi)$ which guarantee existence/uniqueness of $\xi_t$ with pregenerator $G$. All our examples satisfy his conditions.

Notation: let $|\boldsymbol{\xi}|_{\boldsymbol{i}} = \sum_x 1\{\xi(x) = i\}$, $i = 0, 1$

# Some basic questions

Point of view: particle systems are competition models

For a given rate function $c(x, \xi)$, want to determine which if any of the following hold:

# Some basic questions

Point of view: particle systems are competition models

For a given rate function $c(x, \xi)$, want to determine which if any of the following hold:

- **Type $i$ survives**: $|\xi_0|_i \geq 1$ implies $P(|\xi_t|_i \geq 1 \; \forall \; t \geq 0) > 0$

# Some basic questions

Point of view: particle systems are competition models

For a given rate function $c(x, \xi)$, want to determine which if any of the following hold:

- **Type $i$ survives**: $|\xi_0|_i \geq 1$ implies $P(|\xi_t|_i \geq 1 \; \forall \, t \geq 0) > 0$
- **Type $i$ takes over**: $|\xi_0|_i = \infty$ implies

$$P(\xi_t(x) = i \text{ for all large } t) = 1 \quad \forall \, x \in \mathbb{Z}^d$$

# Some basic questions

Point of view: particle systems are competition models

For a given rate function $c(x, \xi)$, want to determine which if any of the following hold:

- **Type $i$ survives**: $|\xi_0|_i \geq 1$ implies $P(|\xi_t|_i \geq 1 \; \forall \, t \geq 0) > 0$
- **Type $i$ takes over**: $|\xi_0|_i = \infty$ implies
$$P(\xi_t(x) = i \text{ for all large } t) = 1 \quad \forall \, x \in \mathbb{Z}^d$$
- **Coexistence:** $\exists$ a stationary distribution $\boldsymbol{\mu}$ for $\xi_t$ s.t.
$$\boldsymbol{\mu}\Big(|\xi|_1 = |\xi|_0 = \infty\Big) = 1$$

## The voter model

Will use throughout:

- $\boldsymbol{p(x)}$ = a symmetric step distribution of irreducible rw on $\mathbb{Z}^d$, $p(0) = 0$, covariance matrix $\sigma^2 I$
- $\boldsymbol{f_i(x, \xi)} = \sum_{y \in \mathbb{Z}^d} p(y - x) 1\{\xi(y) = i\}$ = frequency of type $i$ near $x$ in $\xi$.

# The voter model

Will use throughout:

- $\boldsymbol{p(x)}$ = a symmetric step distribution of irreducible rw on $\mathbb{Z}^d$, $p(0) = 0$, covariance matrix $\sigma^2 I$
- $\boldsymbol{f_i(x, \xi)} = \sum_{y \in \mathbb{Z}^d} p(y - x) 1\{\xi(y) = i\}$ = frequency of type $i$ near $x$ in $\xi$.

Voter model **(neutral competition)**

- Introduced independently: Clifford/Sudbury (1973), Holley/Liggett (1975)
- Flip rate function is $$c_v(x, \xi) = \begin{cases} f_1(x, \xi) & \text{if } \xi(x) = 0 \\ f_0(x, \xi) & \text{if } \xi(x) = 1 \end{cases}$$
- The individual at $x$ dies at rate 1, is replaced by an individual of type $i$ with probability $f_i(x, \xi)$.

## Graphical construction I

- $\boldsymbol{\Lambda^{x,y}}, x, y \in \mathbb{Z}^d$ are independent, rate $p(y-x)$ Poisson processes.

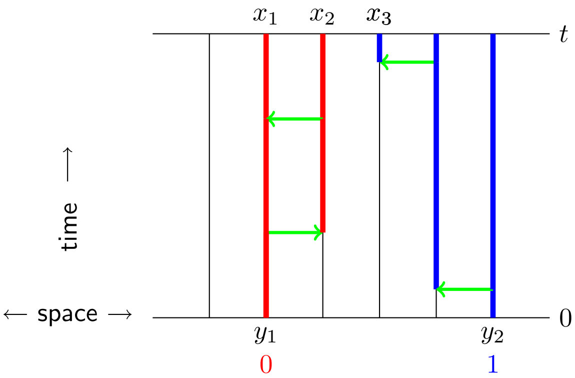
- $\boldsymbol{T_n^{x,y}}, n \geq 1$ are the arrival times of $\Lambda^{x,y}$

- At each time $T_n^{x,y}$
  - **draw an arrow** $\longrightarrow$ from $\boldsymbol{y}$ to $\boldsymbol{x}$, and
  - the voter at $\boldsymbol{x}$ adopts the opinion of the voter at $\boldsymbol{y}$.

- Start with $\xi_0$, determine $\xi_t$ for all $t > 0$.

Note. More complicated $c(x, \xi)$ also have graphical constructions.

# Graphical Construction II



# Coalescing Random Walk Duality

Fix $t > 0$. For each $x \in \mathbb{Z}^d$ let $B_s^{x,t}$, $0 \leq s \leq t$ trace the path **down** and **against** the arrows from $(x, t)$ to $\mathbb{Z}^d \times \{0\}$. Then

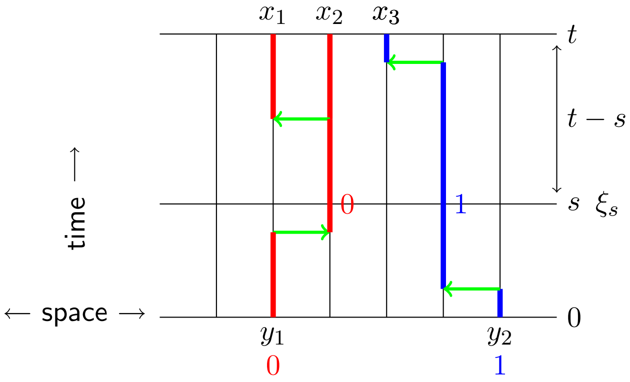
## Coalescing Random Walk Duality

Fix $t > 0$. For each $x \in \mathbb{Z}^d$ let $B_s^{x,t}$, $0 \leq s \leq t$ trace the path **down** and **against** the arrows from $(x, t)$ to $\mathbb{Z}^d \times \{0\}$. Then

- $B_s^{x,t}$ is a rate one random walk with step distribution $p(x)$, $B_0^{x,t} = x$.
- These walks are independent until they meet, at which time they coalesce and move together
- The **duality equation** is: for $0 \leq s \leq t$ and $x \in \mathbb{Z}^d$

$$\boxed{\xi_t(x) = \xi_s(B_{t-s}^{x,t})}$$

# Graphical construction III




$$\xi_t(x) = \xi_s(B^{x,t}_{t-s})$$

Let $\boldsymbol{B_s^x, s \geq 0, x \in \mathbb{Z}^d}$ be a CRW family (note all $s \geq 0$).

**Sample Calculation I.** Assume $d \leq 2$, so rw is recurrent. For any $\xi_0$ and $x \neq y$,

$$\begin{aligned} P(\xi_t(x) \neq \xi_t(y)) &= P(\xi_0(B_t^x) \neq \xi_0(B_t^y)) \\ &\leq P(B_t^x \neq B_t^y) \\ &\to 0 \qquad \text{as } t \to \infty. \end{aligned}$$

So no coexistence for $d \leq 2$.

**Sample calculation 2.** If $\xi_0^u(x)$ are iid Bernoulli($\boldsymbol{u}$), then $\xi_t^u \Rightarrow \xi_\infty^{\boldsymbol{u}}$, whose law is a stationary distribution. $\Rightarrow$ means: for all finite $A, B \subset \mathbb{Z}^d$,

$$\lim_{t\to\infty} P(\xi_t \equiv 1 \text{ on } A, \xi_t \equiv 0 \text{ on } B) \text{ exists}$$

**Sample calculation 2.** If $\xi_0^u(x)$ are iid Bernoulli($\boldsymbol{u}$), then $\xi_t^u \Rightarrow \xi_\infty^{\boldsymbol{u}}$, whose law is a stationary distribution. $\Rightarrow$ means: for all finite $A, B \subset \mathbb{Z}^d$,

$$\lim_{t\to\infty} P(\xi_t \equiv 1 \text{ on } A, \xi_t \equiv 0 \text{ on } B) \text{ exists}$$

Proof. Define the CRW probabilities

- $[\boldsymbol{x}|\boldsymbol{y}]_{\boldsymbol{t}} = P(B_t^x \neq B_t^y)$
- $[\boldsymbol{x}, \boldsymbol{y}|\boldsymbol{z}]_{\boldsymbol{t}} = P(B_t^x = B_t^y \text{ but } \neq B_t^z)$, etc.

**Sample calculation 2.** If $\xi_0^u(x)$ are iid Bernoulli($\boldsymbol{u}$), then $\xi_t^u \Rightarrow \xi_\infty^{\boldsymbol{u}}$, whose law is a stationary distribution. $\Rightarrow$ means: for all finite $A, B \subset \mathbb{Z}^d$,

$$\lim_{t\to\infty} P(\xi_t \equiv 1 \text{ on } A, \xi_t \equiv 0 \text{ on } B) \text{ exists}$$

Proof. Define the CRW probabilities

- $\boldsymbol{[x|y]_t} = P(B_t^x \neq B_t^y)$
- $\boldsymbol{[x,y|z]_t} = P(B_t^x = B_t^y \text{ but } \neq B_t^z)$, etc.

Now calculate

$$P\big(\xi_t(x) = \xi_t(y) = 1, \xi_t(z) = 0\big)$$
$$= P\big(\xi_0(B_t^{x,t}) = \xi_0(B_t^{y,t}) = 1, \xi_0(B_t^{z,t}) = 0\big)$$

**Sample calculation 2.** If $\xi_0^u(x)$ are iid Bernoulli($\boldsymbol{u}$), then $\xi_t^u \Rightarrow \xi_\infty^{\boldsymbol{u}}$, whose law is a stationary distribution. $\Rightarrow$ means: for all finite $A, B \subset \mathbb{Z}^d$,

$$\lim_{t\to\infty} P(\xi_t \equiv 1 \text{ on } A, \xi_t \equiv 0 \text{ on } B) \text{ exists}$$

Proof. Define the CRW probabilities

- $[\boldsymbol{x}|\boldsymbol{y}]_{\boldsymbol{t}} = P(B_t^x \neq B_t^y)$
- $[\boldsymbol{x},\boldsymbol{y}|\boldsymbol{z}]_{\boldsymbol{t}} = P(B_t^x = B_t^y \text{ but } \neq B_t^z)$, etc.

Now calculate

$$\begin{aligned} P\big(\xi_t(x) = &\xi_t(y) = 1, \xi_t(z) = 0\big) \\ &= P\big(\xi_0(B_t^{x,t}) = \xi_0(B_t^{y,t}) = 1, \xi_0(B_t^{z,t}) = 0\big) \\ &= \boldsymbol{u(1-u)}[\boldsymbol{x},\boldsymbol{y}|\boldsymbol{z}]_{\boldsymbol{t}} + \boldsymbol{u^2(1-u)}[\boldsymbol{x}|\boldsymbol{y}|\boldsymbol{z}]_{\boldsymbol{t}} \\ &\to \boldsymbol{u(1-u)}[\boldsymbol{x},\boldsymbol{y}|\boldsymbol{z}]_{\boldsymbol{\infty}} + \boldsymbol{u^2(1-u)}[\boldsymbol{x}|\boldsymbol{y}|\boldsymbol{z}]_{\boldsymbol{\infty}} \quad \text{as } t\to\infty \end{aligned}$$

So, coexistence for $d \geq 3$.

## Martingale problem

Recall

- $\boldsymbol{\Lambda^{x,y}}, x, y \in \mathbb{Z}^d$ are independent, rate $p(y-x)$ Poisson processes.
- $\boldsymbol{T_n^{x,y}}, n \geq 1$ are the arrival times of $\Lambda^{x,y}$
- At each time $T_n^{x,y}$
  - draw an arrow from $y$ to $\boldsymbol{x}$
  - the voter at $\boldsymbol{x}$ adopts the opinion of the voter at $y$.

and restrict to finitely many 1's initially, $|\xi_0|_1 < \infty$.

# Martingale problem

Recall

- $\boldsymbol{\Lambda^{x,y}}, x, y \in \mathbb{Z}^d$ are independent, rate $p(y-x)$ Poisson processes.
- $\boldsymbol{T_n^{x,y}}, n \geq 1$ are the arrival times of $\Lambda^{x,y}$
- At each time $T_n^{x,y}$
  - draw an arrow from $y$ to $\boldsymbol{x}$
  - the voter at $\boldsymbol{x}$ adopts the opinion of the voter at $y$.

and restrict to finitely many 1's initially, $|\xi_0|_1 < \infty$. Then

$$\xi_t(x) = \xi_0(x) + \int_0^t \sum_y \left(\boldsymbol{\xi_{s-}(y) - \xi_{s-}(x)}\right) \Lambda_{x,y}(ds)$$

$$\xi_t(x) = \xi_0(x) + \int_0^t \sum_y \left(\boldsymbol{\xi_{s-}(y) - \xi_{s-}(x)}\right) \Lambda_{x,y}(ds)$$

$$\xi_t(x) = \xi_0(x) + \int_0^t \sum_y \left(\boldsymbol{\xi_{s-}(y) - \xi_{s-}(x)}\right) \Lambda_{x,y}(ds)$$

If $\tilde{\Lambda}_{x,y}(ds) = \Lambda_{x,y}(ds) - p(y-x)ds$, then

$$\xi_t(x) = \xi_0(x) + D_t^x + M_t^x, \text{ where}$$

$$\xi_t(x) = \xi_0(x) + \int_0^t \sum_y \left(\boldsymbol{\xi_{s-}(y) - \xi_{s-}(x)}\right) \Lambda_{x,y}(ds)$$

If $\tilde{\Lambda}_{x,y}(ds) = \Lambda_{x,y}(ds) - p(y-x)ds$, then

$$\xi_t(x) = \xi_0(x) + D_t^x + M_t^x, \text{ where}$$

$$D_t^x = \int_0^t \sum_y \left(\boldsymbol{\xi_s(y) - \xi_s(x)}\right) p(y-x)\, ds$$

$$M_t^x = \int_0^t \sum_y \left(\boldsymbol{\xi_{s-}(y) - \xi_{s-}(x)}\right) \tilde{\Lambda}_{x,y}(ds)$$

$$= \text{ a martingale with square function}$$

$$\langle M^x \rangle_t = \int_0^t \sum_y \left(\boldsymbol{\xi_s(y) - \xi_s(x)}\right)^{\mathbf{2}} p(y-x)\, ds$$

# Measure-valued point of view

Put a unit mass at each 1 of $\xi_t$ to get a measure on $\mathbb{R}^d$,

$X_t = \sum_x \xi_t(x)\delta_x$

For $\phi : \mathbb{R}^d \to \mathbb{R}$ put

$$X_t(\phi) = \sum_x \xi_t(x)\phi(x) = \boldsymbol{X_0(\phi) + D_t(\phi) + M_t(\phi)}$$

# Measure-valued point of view

Put a unit mass at each 1 of $\xi_t$ to get a measure on $\mathbb{R}^d$,
$X_t = \sum_x \xi_t(x)\delta_x$

For $\phi : \mathbb{R}^d \to \mathbb{R}$ put

$$X_t(\phi) = \sum_x \xi_t(x)\phi(x) = \boldsymbol{X_0(\phi) + D_t(\phi) + M_t(\phi)}$$

where $M_t(\phi)$ is a martingale, and (sum by parts)

- $D_t(\phi) = \int_0^t \sum_x \xi_s(x)(p-I)\phi(x)ds = \int_0^t X_s((p-I)\phi)ds$
- $\langle M(\phi)\rangle_t = \int_0^t \sum_x \phi^2(x) \sum_y p(y-x)1\{\xi_x(x) \neq \xi_s(y)\}ds$

Put $\phi \equiv 1$ to get $D_t \equiv 0$, so $|\xi_t|_1$ a nonnegative martingale. Thus no type survives.

## References Ia


### Spin-flip systems, voter model

- Liggett (1985). *Interacting Particle Systems*, Springer-Verlag, New York.

### Super-Brownian Motion, measure-valued diffusions

- Perkins (2002) Measure-valued processes and interactions, in *École d'Été de Probabilités de Saint Flour XXIX-1999*, Lecture Notes Math. **1781**, pages 125-329, Springer-Verlag, Berlin.

### Voter models $\Rightarrow$ super-Brownian motion

- C., Durrett, Perkins (2000) Rescaled voter models converge to super- Brownian motion. *Ann. Probab.* **28** 185-234.

### An application

- C, Perkins (2004) An application of the voter model/super-Brownian motion invariance principle (with E. Perkins). *Ann. Inst. H. Poincaré Probab. Statist.*, **40** 25-32.

# Outline Ib

1. Super-Brownian motion
   - Branching random walk
   - Convergence to super-Brownian motion
2. Voter model convergence to super-Brownian motion
   - Voter model as branching random walk
   - Convergence to super-Brownian motion, $d \geq 3$
   - Sketch of proof

# Branching Random Walk $\eta_t$

System of particles in $\mathbb{Z}^d$

- $p(x)$ as before
- allow multiple particles per site, $\boldsymbol{\eta_t(x)}$ = the number of particles at $x$ at time $t$
- particles at a given site $x$
  - die at rate $\boldsymbol{\delta}$
  - while alive, give birth at rate $\boldsymbol{\beta}$ to a particle which **immediately jumps** to site $y$ with probability $p(y-x)$
- $|\eta_t| = \sum_x \eta_t(x)$ is a cont. time nonspatial branching process
- $\dagger$

# Branching Random Walk $\eta_t$

System of particles in $\mathbb{Z}^d$

- $p(x)$ as before
- allow multiple particles per site, $\boldsymbol{\eta_t(x)}$ = the number of particles at $x$ at time $t$
- particles at a given site $x$
  - die at rate $\boldsymbol{\delta}$
  - while alive, give birth at rate $\boldsymbol{\beta}$ to a particle which **immediately jumps** to site $y$ with probability $p(y-x)$
- $|\eta_t| = \sum_x \eta_t(x)$ is a cont. time nonspatial branching process
- $\dagger$

For a measure-valued point of view, let

- $\mathcal{M}_F$= set of finite Borel measures on $\mathbb{R}$
- $\mu(\phi) = \int_{\mathbb{R}} \phi(x)\mu(dx)$ for $\mu \in \mathcal{M}_F$ and $\phi : \mathbb{R} \to \mathbb{R}$.

# Branching Random Walk $\Rightarrow$ Super-Brownian Motion

**Scale space:** $\boldsymbol{p_N}(x) = p(x\sqrt{N}), \quad x \in \mathbf{S}_N = \mathbb{Z}^d/\sqrt{N}$.

**Scale time:** $\boldsymbol{\eta_t^N(x)}$ has rates

- particles die at rate $\boldsymbol{N + \delta}$
- particles give birth at rate $\boldsymbol{N + \beta}$

**Scale mass:** $\boldsymbol{m_N} = N$ and

$$X_t^N = \frac{1}{\boldsymbol{m_N}} \sum_{x \in \mathbf{S}_N} \eta_t^N(x)\, \delta_x \in \mathcal{M}_F$$

# Branching Random Walk $\Rightarrow$ Super-Brownian Motion

**Scale space:** $\boldsymbol{p_N}(x) = p(x\sqrt{N}), \quad x \in \mathbf{S}_N = \mathbb{Z}^d/\sqrt{N}$.

**Scale time:** $\boldsymbol{\eta_t^N(x)}$ has rates

- particles die at rate $\boldsymbol{N+\delta}$
- particles give birth at rate $\boldsymbol{N+\beta}$

**Scale mass:** $\boldsymbol{m_N} = N$ and

$$X_t^N = \frac{1}{\boldsymbol{m_N}} \sum_{x \in \mathbf{S}_N} \eta_t^N(x)\, \delta_x \in \mathcal{M}_F$$

Expect $X_\cdot^N \Rightarrow$ something as $N \to \infty$. One can check that

With $\boldsymbol{\Delta_N} = N(p_N - I)$, smooth $\phi$, and $\boldsymbol{g} = \beta - \delta$.

$\boldsymbol{X_t^N(\phi) = X_0^N(\phi) + D_t^N(\phi) + M_t^N(\phi)}$, where

With $\boldsymbol{\Delta_N} = N(p_N - I)$, smooth $\phi$, and $\boldsymbol{g} = \beta - \delta$.

$\boldsymbol{X_t^N(\phi) = X_0^N(\phi) + D_t^N(\phi) + M_t^N(\phi)}$, where

$$D_t^N(\phi) = \int_0^t X_s^N(\boldsymbol{\Delta_N}\phi)\,ds + \boldsymbol{g}\int_0^t X_s^N(\phi)\,ds$$
$$\approx \frac{\sigma^2}{2}\int_0^t X_s^N(\boldsymbol{\Delta}\phi)\,ds + \boldsymbol{g}\int_0^t X_s^N(\phi)\,ds$$

With $\boldsymbol{\Delta_N} = N(p_N - I)$, smooth $\phi$, and $\boldsymbol{g} = \beta - \delta$.

$\boldsymbol{X_t^N(\phi) = X_0^N(\phi) + D_t^N(\phi) + M_t^N(\phi)}$, where

$$D_t^N(\phi) = \int_0^t X_s^N(\boldsymbol{\Delta_N}\phi)\,ds + \boldsymbol{g}\int_0^t X_s^N(\phi)\,ds$$
$$\approx \frac{\sigma^2}{2}\int_0^t X_s^N(\boldsymbol{\Delta}\phi)\,ds + \boldsymbol{g}\int_0^t X_s^N(\phi)\,ds$$

$$\langle M^N(\phi)\rangle_t = \frac{1}{N}\int_0^t \sum_y \xi_s^N(y)\sum_x p_N(y-x)(\phi(y)-\phi(x))^2\,ds$$
$$+ (\mathbf{2} + \frac{\boldsymbol{g}}{N})\int_0^t X_s^N(\phi^2)\,ds$$
$$\approx \mathbf{2}\int_0^t X_s^N(\phi^2)\,ds$$

**Theorem**. If $X_0^N \to X_0 \in \mathcal{M}_F$ then $X_\cdot^N \Rightarrow X_\cdot$ as $N \to \infty$, where $X_\cdot$ is SBM$(X_0, 2, \sigma^2, g)$, an $\mathcal{M}_F$-valued processes.

**Theorem**. If $X_0^N \to X_0 \in \mathcal{M}_F$ then $X_\cdot^N \Rightarrow X_\cdot$ as $N \to \infty$, where $X_\cdot$ is SBM$(X_0, 2, \sigma^2, g)$, an $\mathcal{M}_F$-valued processes.

SBM$(X_0, b, \sigma^2, g)$ $X_t$ is characterized† by:

for $\phi \in C_b^3(\mathbb{R})$,

- $X_t(\phi) = X_0(\phi) + \frac{\boldsymbol{\sigma^2}}{2}\int_0^t X_s(\Delta\phi)\,ds + \boldsymbol{g}\int_0^t X_s(\phi) + \boldsymbol{M_t(\phi)}$
- $M_t(\phi)$ is a continuous $L^2$-martingale, with $\langle M(\phi)\rangle_t = \boldsymbol{b}\int_0^t X_s(\phi^2)\,ds$ and
- $\boldsymbol{b} =$ "branching" rate
- $\boldsymbol{\sigma^2} =$ "diffusion" rate
- $\boldsymbol{g} =$ "growth rate"

Measure-valued branching diffusions $X_t, t \geq 0$.

- Introduced independently: Watanabe (1968) and Dawson (1977). ("super-process" name is by Dynkin in 198?)
- Large (!) research literature.
- Many interesting properties, such as: for SBM,

  *For $d \geq 2$, $X_t$ is a.s. supported on a set of zero Lebesgue measure and uniformly spread on its support, in the sense of Hausdorff measure.*

Measure-valued branching diffusions $X_t, t \geq 0$.

- Introduced independently: Watanabe (1968) and Dawson (1977). ("super-process" name is by Dynkin in 198?)
- Large (!) research literature.
- Many interesting properties, such as: for SBM,

  *For $d \geq 2$, $X_t$ is a.s. supported on a set of zero Lebesgue measure and uniformly spread on its support, in the sense of Hausdorff measure.*

Voter model vs. super-Brownian motion?

- Voter model studied since 1975
- SBM studied since 1977
- Some general similarities between the two, but just how closely related can they be?

# Voter model as BRW?

$\xi(x) = 1 \Leftrightarrow$ particle at $x$

$\xi(x) = 0 \Leftrightarrow$ no particle at $x$

Can rephrase the voter dynamics from the **particle point of view**

Recall $f_0(x,\xi) = \sum_y p(y-x) 1\{\xi_t(y) = 0\}$.

A particle at $x$

- dies at rate $\boldsymbol{f_0(x,\xi)}$
- gives birth at rate $\boldsymbol{f_0(x,\xi)}$ to a particle, which jumps to $y$ with probability $p(y-x)1\{\xi(y) = 0\}/f_0(x,\xi)$.
- per particle rates are random

## Voter Model $\Rightarrow$ SBM

- Let $\xi_t^N$ be the rate $\boldsymbol{N}$ voter model on $\mathbf{S}_N = \mathbb{Z}^d/\sqrt{N}$.
- $\gamma_e = \sum_y p(y)[0|y]_\infty$
- $X_t^N = \dfrac{1}{m_N} \displaystyle\sum_{x \in \mathbf{S}_N} \xi_t^N(x)\delta_x,$ $\qquad (m_N = N$ for $d \geq 3)$.

## Voter Model $\Rightarrow$ SBM

- Let $\xi_t^N$ be the rate $\boldsymbol{N}$ voter model on $\mathbf{S}_N = \mathbb{Z}^d/\sqrt{N}$.
- $\gamma_e = \sum_y p(y)[0|y]_\infty$
- $X_t^N = \frac{1}{m_N} \sum_{x \in \mathbf{S}_N} \xi_t^N(x)\delta_x$,   $(m_N = N$ for $d \geq 3)$.

**Theorem (C,Durrett, Perkins 2000)**

*Assume* $d \geq 3$, $|\xi_0^N| \leq CN$ *and* $X_0^N \to X_0$. *Then* $X_\cdot^N \Rightarrow X_\cdot$ *as* $N \to \infty$ *where* $X_\cdot$ *is SBM*$(X_0, 2\gamma_e, \sigma^2, 0)$.

This is a low density result. It describes the behavior of the voter model when 1's are relatively sparse.

- $\mathbb{Z}^d/\sqrt{N}$ has $\boldsymbol{N^{d/2}}$ sites/volume, but
- $|\boldsymbol{\xi_t^N}|_1 = \boldsymbol{O(N)}$ $(d \geq 3)$
- Consistent with behavior of $\mathrm{supp}$(SBM).

This is a low density result. It describes the behavior of the voter model when 1's are relatively sparse.

- $\mathbb{Z}^d/\sqrt{N}$ has $\boldsymbol{N^{d/2}}$ sites/volume, but
- $|\boldsymbol{\xi_t^N}|_1 = \boldsymbol{O(N)}$ $(d \geq 3)$
- Consistent with behavior of $\mathrm{supp}$(SBM).

Application: can use this to give a “simpler” proof of a result of Sawyer (1977) (which has an amazing proof).

This is a low density result. It describes the behavior of the voter model when 1's are relatively sparse.

- $\mathbb{Z}^d/\sqrt{N}$ has $\boldsymbol{N^{d/2}}$ sites/volume, but
- $|\boldsymbol{\xi_t^N}|_1 = \boldsymbol{O(N)}$ ($d \geq 3$)
- Consistent with behavior of $\mathrm{supp}$(SBM).

Application: can use this to give a "simpler" proof of a result of Sawyer (1977) (which has an amazing proof).

Our proof of voter model $\Rightarrow$ SBM:

1. establish tightness by verifying Jakubowski's conditions (see Perkins (2002))
2. show all subsequential limits of $X_\cdot^N$ satisfy SBM martingale problem with the claimed parameters.

## In more detail . . .

$\boldsymbol{A_N \approx B_N}$ means $\boldsymbol{E|A_N - B_N|^p \to 0}$, some $p \geq 1$.

Recall

$$X_t^N(\phi) = \frac{1}{m_N} \sum_x \xi_t^N(x)\phi(x) = \boldsymbol{X_0^N(\phi) + D_t^N(\phi) + M_t^N(\phi)}$$

1. The drift term is: with $\boldsymbol{\Delta_N} = N(p_N - I)$,

$$D_t^N(\phi) = \frac{\boldsymbol{N}}{m_N} \int_0^t \sum_x \xi_s^N(x) \sum_{\boldsymbol{y}} \boldsymbol{p_N(y-x)(\phi(y)-\phi(x))}ds$$

$$= \int_0^t X_s^N(\boldsymbol{\Delta_N \phi})ds$$

$$\approx \frac{\sigma^2}{2} \int_0^t X_s^N(\boldsymbol{\Delta \phi})ds \qquad \checkmark$$

2. The martingale square function

$$\begin{aligned}
\langle M^N(\phi)\rangle_t &= \int_0^t \frac{1}{N}\sum_x \phi^2(x)\sum_y p_N(y-x)1\{\xi_s^N(x) \neq \xi_s^N(y)\})ds \\
&\approx 2\int_0^t \frac{1}{N}\sum_x \phi^2(x)\sum_y p_N(y-x)\xi_s^N(x)(1-\xi_s^N(y))ds \\
&= 2\int_0^t m_N(s)ds
\end{aligned}$$

2. The martingale square function

$$\langle M^N(\phi)\rangle_t = \int_0^t \frac{1}{N}\sum_x \phi^2(x)\sum_y p_N(y-x)1\{\xi_s^N(x)\neq\xi_s^N(y)\})ds$$

$$\approx 2\int_0^t \frac{1}{N}\sum_x \phi^2(x)\sum_y p_N(y-x)\xi_s^N(x)(1-\xi_s^N(y))ds$$

$$= 2\int_0^t m_N(s)ds$$

Let $t_N \downarrow 0$ with $Nt_N \to \infty$, for $s > t_N$ put $s' = s - t_N$. Let $\hat{E}_N$ be law of rate $N$ CRW's.

2. The martingale square function

$$\langle M^N(\phi)\rangle_t = \int_0^t \frac{1}{N}\sum_x \phi^2(x)\sum_y p_N(y-x)1\{\xi_s^N(x)\neq\xi_s^N(y)\}ds$$

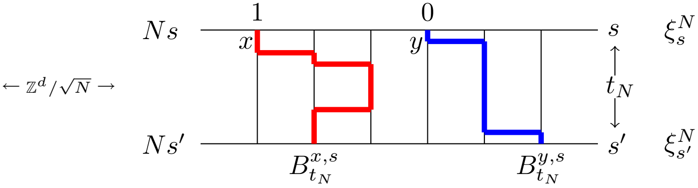
$$\approx 2\int_0^t \frac{1}{N}\sum_x \phi^2(x)\sum_y p_N(y-x)\xi_s^N(x)(1-\xi_s^N(y))ds$$

$$= 2\int_0^t m_N(s)ds$$

Let $t_N \downarrow 0$ with $Nt_N \to \infty$, for $s > t_N$ put $s' = s - t_N$. Let $\hat{E}_N$ be law of rate $N$ CRW's.

**Step 1** $\int_0^t m_N(s)ds \approx \int_{t_N}^t E(m_N(s) \mid \mathcal{F}_{s'})ds$ ✓

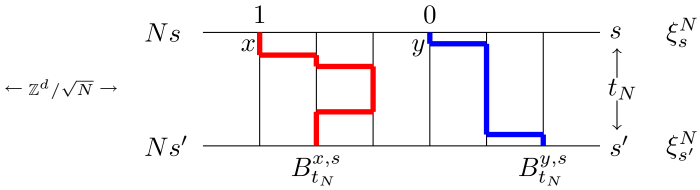
**Step 2** $E(m_N(s) \mid \mathcal{F}_{s'}) \approx \gamma_e X_{s'}^N(\phi)$




$$E(\xi^N_s(x)(1-\xi^N_s(y) \mid \mathcal{F}_{s'}))$$

$$= \hat{E}^N(\xi^N_{s'}(B^x_{t_N})(1-\xi^N_{s'}(B^y_{t_N}))) \qquad \text{duality}$$

$$\approx \hat{E}^N(\boldsymbol{\xi^N_{s'}(B^x_{t_N})} \mathbf{1}\{\boldsymbol{B^x_{t_N} \neq B^y_{t_N}}\}) \qquad \text{sparse } 1's \text{ in } \xi^N_{s'}$$




$$E(\xi^N_s(x)(1-\xi^N_s(y) \mid \mathcal{F}_{s'}))$$

$$= \hat{E}^N(\xi^N_{s'}(B^x_{t_N})(1-\xi^N_{s'}(B^y_{t_N}))) \qquad \text{duality}$$

$$\approx \hat{E}^N(\boldsymbol{\xi^N_{s'}(B^x_{t_N})}\mathbf{1}\{\boldsymbol{B^x_{t_N} \neq B^y_{t_N}}\}) \qquad \text{sparse } 1's \text{ in } \xi^N_{s'}$$

$$E(m_N(s) \mid \mathcal{F}_{s'})$$

$$\approx \frac{1}{N}\sum_x \phi^2(x) \sum_y p_N(y-x)\left[\quad \downarrow \quad\right]$$

$$\approx \frac{1}{N}\sum_x \phi^2(x)\boldsymbol{\xi^N_{s'}(x)}\sum_y p(y-x)\boldsymbol{[x|y]_{Nt_N}} \qquad \phi \text{ cont.}, B^x_{t_N} \approx x$$

$$= \gamma^N_e X^N_{s'}(\phi^2), \qquad \text{where } \gamma_e = \sum_e p(e)[0|e]_\infty \qquad \checkmark$$