Convergence of Rescaled Competing Species Processes to a Class of SPDEs

Sandra Kliem

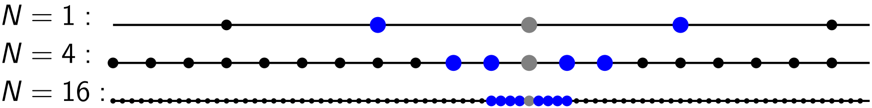
EURANDOM

YEP - March 17, 2011

A class of rescaled competing species processes

We define a sequence $\xi_t^N, N \in \mathbb{N}$ of rescaled competing species models, which can be described as perturbations of rescaled voter models.
In the N^{th} model:

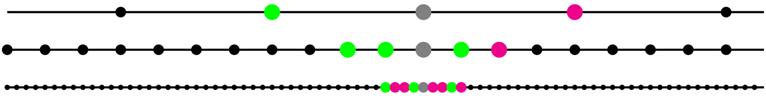
- space: $\mathbb{Z}/N$,
- neighbours of x: $y \sim x$ iff $0 < |x - y| \leq N^{-1/2}$



Each x has $2c(N)N^{1/2}$, $c(N) \overset{N\to\infty}{\to} 1$ neighbours.

Long-range interaction takes into account the densities of the neighbours of $x \in \mathbb{Z}/N$ at long-range, i.e.

$$f_i^{(N)}(x,\xi) \equiv \frac{1}{|y : y \sim x|} \sum_{y:y\sim x} 1(\xi^N(y) = i), \quad i = 0, 1.$$



Note in particular:

- $0 \leq f_i^{(N)} \leq 1$ and
- $f_0^{(N)} + f_1^{(N)} = 1.$

Recall:

- Flip rates of the *unscaled biased voter process*:

$$0 \to 1 \text{ at rate } c(x,\xi) = (1+\tau) f_1(x,\xi),$$
$$1 \to 0 \text{ at rate } c(x,\xi) = f_0(x,\xi).$$

- Rescaling for the *biased voter process*::

$$\begin{aligned} 0 \to 1 \text{ at rate } c(x,\xi) &= N\left(1+\frac{\tau}{N}\right) f_1^{(N)}(x,\xi) \\ &= N f_1^{(N)}(x,\xi) + f_1^{(N)}(x,\xi)\tau, \\ 1 \to 0 \text{ at rate } c(x,\xi) &= N f_0^{(N)}(x,\xi). \end{aligned}$$

- Adding more general perturbations:

$$0 \to 1 \text{ at rate } N f_1^{(N)} + f_1^{(N)} G_0^{(N)}\left(f_1^{(N)}\right),$$
$$1 \to 0 \text{ at rate } N f_0^{(N)} + f_0^{(N)} G_1^{(N)}\left(f_0^{(N)}\right),$$

where $G_i^{(N)}, i = 0, 1$ are power series on $[0,1]$, i.e.

$$G_i^{(N)}(x) = \sum_{m=0}^{\infty} \alpha_i^{(m+1,N)} x^m, \quad i = 0, 1, x \in [0, 1]$$

with $\alpha_i^{(m+1,N)}$ satisfying certain summability and convergence conditions, uniformly in $N \geq N_0$. Define

$$G_i(x) \equiv \lim_{N\to\infty} G_i^{(N)}(x) = \sum_{m=0}^{\infty} \alpha_i^{(m+1)} x^m, \quad x \in [0, 1].$$

The object of interest

Approximate density $A(\xi_t^N)$ for the configurations ξ_t^N:

$$A(\xi_t^N)(x) = \frac{1}{|y : y \sim x|} \sum_{y \sim x} \xi_t^N(y), \qquad x \in \mathbb{Z}/N.$$

Note: $A(\xi_t^N)(x) = f_1^{(N)}\left(x, \xi_t^N\right)$.

By linearly interpolating between sites we obtain approximate densities $A(\xi_t^N)(x) \in [0,1]$ for all $x \in \mathbb{R}$.

Notation

Set $\mathcal{C}_1 \equiv \{f : \mathbb{R} \to [0,1]$ *continuous*$\}$ *and let* $\mathcal{C}_1$ *be equipped with the topology of uniform convergence on compact sets.*

We obtain that $t \mapsto A(\xi_t^N)$ is cadlag $\mathcal{C}_1$-valued, i.e. $A(\xi_\cdot^N) \in D(\mathcal{C}_1)$.

Theorem

Suppose that $A(\xi_0^N) \to u_0$ *in* $\mathcal{C}_1$ *and that* $G_i^{(N)}$, $i = 0, 1$ *satisfy appropriate Hypotheses. Then*

- $\big(A(\xi_t^N) : t \geq 0\big)$ *are C-tight as cadlag* $\mathcal{C}_1$*-valued processes.*
- *The limit points of* $A(\xi_t^N)$ *are continuous* $\mathcal{C}_1$*-valued processes* u_t *which solve*

$$\frac{\partial u}{\partial t} = \frac{\Delta u}{6} + (1-u)u\left\{G_0(u) - G_1(1-u)\right\} + \sqrt{2u(1-u)}\dot{W}$$

  with initial condition u_0.

- *If we assume additionally* $\int u_0(x)dx < \infty$, *then* u_t *is the unique in law* $[0,1]$*-valued solution to the above SPDE.*

Example

Consider a sequence of rescaled Lotka-Volterra models with rates of change

$$0 \to 1 \text{ at rate } Nf_1^{(N)} \left(f_0^{(N)} + a_{01}^{(N)} f_1^{(N)} \right),$$

$$1 \to 0 \text{ at rate } Nf_0^{(N)} \left(f_1^{(N)} + a_{10}^{(N)} f_0^{(N)} \right).$$

For $i = 0, 1$ *choose*

$$a_{i(1-i)}^{(N)} - 1 \equiv \frac{\theta_i^{(N)}}{N} \text{ with } \theta_i^{(N)} \stackrel{N\to\infty}{\to} \theta_i$$

and rewrite

$$0 \to 1 \text{ at rate } Nf_1^{(N)} + \theta_0^{(N)} \left(f_1^{(N)} \right)^2 = Nf_1^{(N)} + f_1^{(N)} \theta_0^{(N)} f_1^{(N)},$$

$$1 \to 0 \text{ at rate } Nf_0^{(N)} + \theta_1^{(N)} \left(f_0^{(N)} \right)^2 = Nf_0^{(N)} + f_0^{(N)} \theta_1^{(N)} f_0^{(N)}.$$

The Theorem yields that the sequence of approximate densities $A(\xi_t^N)$ is tight and every solution solves

$$\frac{\partial u}{\partial t} = \frac{\Delta u}{6} + (1-u)u\{\theta_0 u - \theta_1(1-u)\} + \sqrt{2u(1-u)}\dot{W}$$

with initial condition u_0. Uniqueness in law holds for initial conditions of finite mass.

Literature Review

- This paper is an extension of results of Mueller and Tribe [3] ($d = 1$, voter processes with nonnegative bias).
- In Cox and Perkins [1] it was shown that rescaled Lotka-Volterra models with long-range interaction converge weakly to super-Brownian motion with linear drift. They consider
  - low density regime
  - **weak** limits for measure-valued processes

$$X_t^N = \frac{1}{N} \sum_{x \in \mathbb{Z}/(M_N\sqrt{N})} \xi_t^N(x)\delta_x$$

  with $M_N/\sqrt{N} \to \infty$ (for $d = 1$)
  - We consider $M_N = \sqrt{N}$ (we get X_t^N converges to $u_t dt$ in the **vague** topology).

Ideas used in the Proof

Part 1: "How to get positive perturbations only"

Rewrite the rates in a form, where all resulting coefficients are non-negative by using

$$-x^m = (1-x)\sum_{l=1}^{m-1} x^l - x \quad \text{and} \quad 1 - f_1 = f_0.$$

Part 2: Tightness

Graphical construction

Suppose

$$0 \to 1 \text{ at rate } \cdots + q_j^{(0,m)} f_j f_1^{m-1} + \cdots$$

with $j \in \{0,1\}$, $q_j^{(0,m)} > 0$.

Recall: $|y : y \sim x| = 2c(N)\sqrt{N}$ and
$f_i^{(N)}(x,\xi) \equiv \frac{1}{2c(N)\sqrt{N}} \sum_{y:y\sim x} 1(\xi^N(y) = i)$, $i = 0,1$.

The graphical construction uses independent families of i.i.d. Poisson processes: E.g.,

$$\left(Q_t^{m,j,0}(x; y_1, \dots, y_m) : x, y_1, \dots, y_m \in N^{-1}\mathbb{Z} \right)$$

$$\text{i.i.d. Poisson processes of rate } \frac{q_j^{(0,m)}}{2c(N)\sqrt{N}(2c(N)\sqrt{N})^{m-1}}.$$

At a jump of $Q_t^{m,j,0}(x; y_1, \dots, y_m)$ the voter at x adopts the opinion 1 provided that $y_1, \dots, y_m$ are neighbours of x, y_1 has opinion j and all of $y_2 \dots, y_m$ have opinion 1.

- Graphical construction
  $\Rightarrow$ stochastic integral equation for ξ_t^N
- integrate against test-function $\phi_t(x)$
  $\Rightarrow$ an approximate semimartingale decomposition
- choose "clever" test function
  $\Rightarrow$ approximate Green's function representation for $A(\xi_t)$.

Tightness estimates

Derive estimates on p^{th}-moment differences, i.e. bound (I omit some details here)

$$\mathbb{E}\left[\left|A(\xi_t^N)(z) - A(\xi_s^N)(y)\right|^p\right] \leq Ce^{\lambda p|z|}\left(|t-s|^{p/24} + |z-y|^{p/24} + N^{-p/24}\right).$$

Then use Kolmogorov's continuity theorem and the Arzelà-Ascoli theorem.

Part 3: Uniqueness in law

Apply a version of Dawson's Girsanov theorem.

References

COX, J.T. and PERKINS, E.A. Rescaled Lotka-Volterra models converge to super-Brownian motion. *Ann. Probab.* (2005)

COX, J.T. and PERKINS, E.A. Survival and coexistence in stochastic spatial Lotka-Volterra models. *Probab. Theory Related Fields* (2007)

MUELLER, C. and TRIBE, R. Stochastic p.d.e.'s arising from the long range contact and long range voter processes. *Probab. Theory Related Fields* (1995)

thank you