Asymptotic behavior of the survival probability for a critical branching process in markovian environment

Yinna YE

Workshop YEP VIII 2011
EURANDOM

17 March 2011

Consider

- a measurable space (E, ξ) and its corresponding product measurable space $(\Omega, \mathfrak{F}) = (E^{\mathbb{N}}, \xi^{\mathbb{N}})$;
- a probability law Π on Ω
- a family $(p_\theta)_{\theta \in E}$ of probability law p_θ on $\mathbb{N}$; we denote by g_θ the generating function of p_θ defined by

$$g_\theta(s) := \sum_{k=0}^{+\infty} p_\theta(k) s^k, \text{ for } 0 \le s \le 1.$$

- Any element $\omega = (\omega_i)_{i \ge 0} \in \Omega$ is called an **environment process**.

- For a fixed environment process $\omega = (\omega_i)_{i\geq 0} \in \Omega$, we consider a branching process $(Z_n)_{n\geq 0}$ such that $Z_0 = 1$ and the reproduction law of an individual of the generation i has the generating function g_i.
  The generating function of Z_n is

$$G_n(s) := g_0 \circ g_1 \circ \ldots \circ g_{n-1}(s) \quad 0 \leq s < 1.$$

- Denote by $\mathbb{P}_\omega$ the conditional probability of the branching process $(Z_n)_n$ on $\Lambda = \mathbb{N}^{\mathbb{N}}$, given the environment $\omega \in \Omega$.
- The total probability denoted by $\mathbb{P}$, defined on $\Lambda \times \Omega$, is defined by

$$\mathbb{P} = \int_\Omega \mathbb{P}_\omega \otimes \delta_\omega \mathrm{d}\Pi(\omega).$$

We denote by $\mathbb{E}$ the corresponding expectation.

In the case of a stationary ergodic environment (ie Π is invariant and ergodic for the shift on Ω), we have the following results:

Theorem (Athreya & Karlin, 1971)

1. *If* $\mathbb{E}[\ln g_0'(1)] \leq 0$, *then*

$$\mathbb{P}_\omega(\lim_{n\to+\infty} Z_n = 0) = 1, \quad \Pi - a.s.$$

2. *If* $\mathbb{E}[\ln g_0'(1)] > 0$ *and* $\mathbb{E}\{-\ln(1-g_0(0))\} < +\infty$, *then*

$$\mathbb{P}_\omega(\lim_{n\to+\infty} Z_n = 0) < 1, \quad \Pi - a.s.$$

The branching process $(Z_n)_{n\geq 0}$ is called

- **supercritical**, if $\mathbb{E}[\ln g_0'(1)] > 0$;
- **critical**, if $\mathbb{E}[\ln g_0'(1)] = 0$;
- **subcritical**, if $\mathbb{E}[\ln g_0'(1)] < 0$.

In the case of i.i.d. environment (ie $\Pi = \mu^{\otimes \mathbb{N}}$ for some probability μ on E), we have

Theorem (Guivarc'h, Le Page & Liu 2003)

if $\mathbb{E}[\ln g_0'(1)] = 0$, $0 < \mathbb{E}[\ln g_0'(1)]^2 < +\infty$ *and* $\mathbb{E}\{[\frac{g_0''(1)}{(g_0'(1))^2}]^\varepsilon\} < +\infty$ *for some* $\varepsilon > 0$, *then*

$$\mathbb{P}(Z_n > 0) \sim \frac{C}{\sqrt{n}}, \quad \text{as } n \to +\infty,$$

where $C \in \mathbb{R}^{*+}$.

Let

- $\mathcal{X}$ be a finite set
- $X = (X_n)_{n\geq 0}$ be an irreducible and aperiodic Markov chain on $\mathcal{X}$ with transition matrix $P = (p_{i,j})_{i,j\in\mathcal{X}}$;
- ν be the (unique) P-invariant probability measure on $\mathcal{X}$.

We denote by G the semi-group of all generating functions of probability measures on $\mathbb{N}$ and $\mathcal{G}$ its σ-algebra.
We consider a finite family $(\overline{F}(i,j,\cdot))_{i,j\in\mathcal{X}}$ of probabilities on $(G,\mathcal{G})$.

- Consider now the Markov chain $(M_n)_{n\geq 0} = (g_n, X_n)_{n\geq 0}$ with values in $G \times \mathcal{X}$ with transition probability Q

$$Q\{(g,i),(A\times\{j\})\} = p_{i,j}\overline{F}(i,j,A).$$

- The Markov chain $(M_n)_{n\geq 0}$ will be our **environment process**.
- Given the environment $(M_n)_{n\geq 0}$, we consider the branching process $(Z_n)_{n\geq 0}, Z_0 = 1$ associated to the sequence $(g_n)_{n\geq 0}$.

Note that the g_n are in "markovian dependance"; in the case when $\mathcal{X}$ reduces to one point, the random environment is i.i.d.

Goal: to generalize Guivarc'h, Le Page & Liu's theorem [2003] in the case when $(Z_n)_{n\geq 0}$ is in markovian environment.

Consider $h : G \to \overline{\mathbb{R}}_+$, $g \mapsto h(g) := \ln g'(1)$. The image of the probability $\overline{F}(i,j,\cdot)$ by the map h is denoted by $F(i,j,\cdot)$. Assume that the following hypotheses (H) are satisfied:

- H1 there exist $\alpha > 0$, such that for all $\lambda \in \mathbb{C}$ satisfying $|\mathrm{Re}\lambda| \le \alpha$, we have
$$\sup_{(i,j)\in\mathcal{X}\times\mathcal{X}} |\widehat{F}(i,j,\lambda)| < +\infty,$$
where $\widehat{F}(i,j,\lambda) = \int_{\mathbb{R}} e^{\lambda t} F(i,j,\mathrm{d}t)$;
- H2 there exist $n_0 \ge 1$ and $(i_0,j_0) \in \mathcal{X}\times\mathcal{X}$ such that the measure $\mathbb{P}_{i_0}(X_{n_0} = j_0, S_{n_0} \in \mathrm{d}x)$ has an absolutely continuous component with respect to the Lebesgue measure on $\mathbb{R}$;
- H3 $$\sum_{(i,j)\in\mathcal{X}\times\mathcal{X}} \nu_i p_{i,j} \int_{\mathbb{R}} tF(i,j,\mathrm{d}t) = 0.$$

Theorem

Under hypotheses (H), for any $(i,j) \in \mathcal{X} \times \mathcal{X}$, *there exists a constant* $0 < \beta_{i,j} < +\infty$ *such that*

$$\mathbb{P}(Z_n > 0, X_n = j / M_0 = (Id, i)) \sim \frac{\beta_{i,j}}{\sqrt{n}}, \quad \text{as } n \to +\infty.$$

- Given the environment $(M_n)_{n\geq 0}$, the survival probability of the branching process $(Z_n)_{n\geq 0}$ at the generation n is equal to

$$q_n := 1 - G_n(0).$$

- Setting $S_0 = 0$ and $S_n = S_0 + Y_1 + \cdots + Y_n$ with $Y_n = \ln g'_{n-1}(1)$ for $n \geq 1$, one gets:

$$q_n^{-1} = \exp(-S_n) + \sum_{k=0}^{n-1} \eta_{k,n} \exp(-S_k),$$

where, for $0 \leq k \leq n-1$ and $s \in [0, 1[$

1. $\eta_{k,n} = f_k(g_{k+1,n}(0))$;
2. $f_k(s) = \dfrac{1}{1 - g_k(s)} - \dfrac{1}{g'_k(1)(1-s)}$;
3. $g_{k,n} = g_k \circ g_{k+1} \circ \cdots \circ g_{n-1}$ and $g_{n,n} = Id$;

Set $m_n = \min(S_0, S_1, \cdots, S_n)$.

Theorem (Local limit theorem)

Under the hypotheses (H), for all $(i, j) \in \mathcal{X} \times \mathcal{X}$, one gets

$$\lim_{n \to +\infty} \sqrt{n}\, \mathbb{P}_i(m_n \geq -x, X_n = j) = h_{i,j}(x), \tag{1}$$

where the functions $(x, i) \mapsto h_{i,j}(x)$ are harmonic for $(S_n, X_n)_{n \geq 0}$ and satisfy

- *for any $i, j \in E$, $x \mapsto h_{i,j}$ is increasing;*
- *$h_{i,j}(x) > 0$ for $x \geq 0$.*

Moreover,

$$h_{i,j}(x) \sim x \sqrt{\frac{2}{\sigma^2}}\, \nu_j, \quad \text{as } x \to +\infty.$$

The proof of the local limit theorem is based on

1. the factorization theory of Presman;
2. several technics from the theory of complex variable functions (analytic continuation, Weiertrass preparation lemma, residue theorem ...) ...

- We want to check that

$$\sqrt{n}\,\mathbb{P}_i(Z_n > 0) \overset{n\to+\infty}{\longrightarrow} \beta_i, \tag{2}$$

whith $\beta_i > 0$.

- Since

$$\mathbb{P}_i(Z_n > 0) = \mathbb{P}_i(Z_n > 0, m_n < -x) + \mathbb{P}_i(Z_n > 0, m_n \geq -x),$$

the equality (2) is an immediate consequence of the following two lemmas

Lemma 1

Under the conditions (H) , we have

$$\lim_{n\to+\infty} \sqrt{n}\, \mathbb{P}_i(Z_n > 0, m_n \geq -x) = \beta_i(x),$$

where $\beta_i(x) > 0$, *for any* $i \in \mathcal{X}$. *Moreover,* $\lim_{x\to+\infty} \beta_i(x) = \beta_i > 0$.

Lemma 2

Under the conditions (H) , we have for any $n \geq 0$,

$$0 \leq \mathbb{P}_i(Z_n > 0, m_n < -x) \leq \theta(x)$$

where θ *satisfies* $\lim_{x \to +\infty} \theta(x) = 0$.

Thank you for your attention!

yinna.ye@lmpt.univ-tours.fr