Frogs in a random environment

Lorenz Pfeifroth

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Gliederung

1 Frog Model in a fixed environment
- Definition
- Results

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On every vertex we have some sleeping frogs. The frog at the origin starts to jump according to a random walk.If an active frog visits a vertex for the first time, the sleeping frogs wake up and move independently of all other active frogs according to the same random walk.

Let $G = (\mathbb{Z}, E)$, $U := [0,1]^{\mathbb{Z}}$ and

$$\mathcal{S} := \{\eta = (\eta_i)_{i \in \mathbb{Z}} \in \mathbb{N}^{\mathbb{Z}} : \eta_0 = 1\}.$$

Let $\Omega \subset \mathbb{Z}^{\mathbb{N}_0}$ denote the set of all paths in G. Let $\mathcal{F}$ be the canonical σ-algebra on Ω and $X_t : \Omega \to \mathbb{Z}$ the canonical projections to the t-th coordinate.

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Let $\alpha \in U$ fixed. Q_α is the probability measure on $(\Omega, \mathcal{F})$ of a Markov chain such that for all $t \in \mathbb{N}_0$ and $n \in \mathbb{Z}$, one has

$$Q_\alpha(X_{t+1} = n+1 | X_t = n) = \alpha_n,$$
$$Q_\alpha(X_{t+1} = n-1 | X_t = n) = 1 - \alpha_n := \beta_n.$$

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Definition (Recurrence for FM_α)

The frog model $FM_\alpha(\mathbb{Z}, \eta)$ is called recurrent if and only if the P_η-probability that the origin is visited infinitely often is equal 1. Otherwise the frog model is called transient.

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Theorem (Gantert and Schmidt 2009)

Let $p \in (0.5, 1)$ and η be fixed. Then the frog model $FM_p(\mathbb{Z}, \eta)$ is recurrent if and only if

$$\sum_{i=1}^{\infty} \eta_i \rho^i = \infty,$$

with $\rho = \frac{p}{1-p}$.

For the next result, we will take $\eta = (\eta_i)_{i \in \mathbb{Z}}$ as random variable with values in $\mathcal{S}$. We always assume that:

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1. η_i for $i \neq 0$ are iid random variable.
2. We denote the distribution of η by μ.

Remark: $\mu(\eta_1 \geq 1) = 1$.

Theorem (Gantert and Schmidt 2009)

Let $p \in (0.5, 1)$ and η_i are iid random variables. Then μ-a.s.

$$P_\eta(0 \text{ is visited i.o}) = \begin{cases} 0 & \text{if} \quad \mathbb{E}[\log^+(\eta_1)] < \infty \\ 1 & \text{if} \quad \mathbb{E}[\log^+(\eta_1)] = \infty \end{cases}.$$

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2. $$\rho_i = \begin{cases} \sigma_1 \cdots \sigma_i & \text{if } i \geq 1 \\ \sigma_0 \cdots \sigma_i & \text{if } i \leq 0 \end{cases}.$$

3. We call an environment $\alpha \in U$ irreducible if and only if $0 < \alpha_i < 1$ for all $i \in \mathbb{Z}$.

Theorem (P. 2010)

Let $\alpha \in U$ and $\eta \in \mathcal{S}$ be fixed. Also assume

$$\sum_{n=0}^{\infty} (\rho_{-n})^{-1} = \infty \text{ and } \sum_{n=1}^{\infty} \rho_n < \infty$$

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$$\sum_{n=0}^{\infty} (\rho_{-n})^{-1} = \infty \text{ and } \sum_{n=1}^{\infty} \rho_n < \infty$$

and let α be irreducible. Then the frog model $FM_\alpha(\mathbb{Z}, \eta)$ is recurrent if and only if

$$\sum_{i=1}^{\infty} \eta_i \sigma_i^{(S)} = \infty$$

with $\sigma_i^{(S)} := \sum_{n=i}^{\infty} \sigma_1 \cdots \sigma_n = \sum_{n=i}^{\infty} \rho_n$.

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$$A' := \{\eta; \text{there exists } K(\eta) \in \mathbb{R} \textit{ such that for all } x \leq 1 \quad \sum_{i=1}^{\infty} \eta_i \sigma_{x,i}^{(S)} < K(\eta)\}$$

with $\sigma_{x,i}^{(S)} := \sum_{n=i+x}^{\infty} \sigma_x \cdots \sigma_n$.

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with $\sigma_{x,i}^{(S)} := \sum_{n=i+x}^{\infty} \sigma_x \cdots \sigma_n$*.If* $\mu(A') = 1$*, then we get* μ*-a.s.*

$$P_\eta(\text{the origin is visited only finitely often by active frogs}) = 1.$$

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$$P'_\eta = \int_U P_{\eta,\alpha} \, \mathbb{P}(d\alpha).$$

Thus, P'_η is the probability measure of a new model $(FM_{\mathbb{P}}(\mathbb{Z}, \eta))$ called frog model in a random environment.

Definition (Recurrence for $FM_{\mathbb{P}}$)

The frog model $FM_{\mathbb{P}}(\mathbb{Z}, \eta)$ is called recurrent if and only if the P'_η-probability that the origin is visited infinitely often is equal 1. Otherwise the frog model is called transient.

For all the following results we make the following assumptions

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That means the underlying random walk is transient to the right.

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$$\sum_{i=1}^{\infty} \eta_i e^{i(\mathbb{E}[\log(\sigma_0)]-\epsilon)} = \infty,$$

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2. *If there is a $\epsilon > 0$ with $\epsilon < -\mathbb{E}_{\nu}[\log(\sigma_0)]$ and*

$$\sum_{i=1}^{\infty} \eta_i e^{i(\mathbb{E}[\log(\sigma_0)]+\epsilon)} < \infty,$$

   then $FM_{\mathbb{P}}(\mathbb{Z}, \eta)$ is transient.

Now let

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Also we denote $R \geq 0$ the radius of convergence of f_η.

Corollary (P. 2010)

Let $\eta \in \mathcal{S}$ *be fixed and* $\mathbb{E}[|\log(\sigma_0)|] < \infty$*.*

1. *If* $\mathbb{E}[\log(\sigma_0)] \neq 0$ *and* $R < \exp(-|\mathbb{E}[\log(\sigma_0)]|)$ *then* $FM_{\mathbb{P}}(\mathbb{Z}, \eta)$ *is recurrent.*

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3. *If* $\mathbb{E}[\log(\sigma_0)] = 0$ *then* $FM_{\mathbb{P}}(\mathbb{Z}, \eta)$ *is recurrent.*

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Theorem (P. 2010)

Let η_i, $i \in \mathbb{Z} \setminus \{0\}$, be iid. Then μ-a.s.

$$P'_\eta(FM_{\mathbb{P}}(\mathbb{Z}, \eta) \text{ is recurrent}) = \begin{cases} 0 & \text{if} \quad E_\mu[\log^+(\eta_1)] < \infty \\ 1 & \text{if} \quad E_\mu[\log^+(\eta_1)] = \infty \end{cases}.$$

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In particular, if the distribution of η_1 is given the distribution of the environment doesn't change the probability to be recurrent.

For $\mathbb{P}$-a.a α there exists a K_α such that $\mathbb{P}$-a.s for all $x \in \mathbb{Z}^-$

$$\sum_{k=x}^{\infty} \sigma_x \cdots \sigma_k \leq K_\alpha.$$

Theorem (P. 2011)

Assume that η_i, $i \in \mathbb{Z} \setminus \{0\}$, are iid and α_i, $i \in \mathbb{Z}$, are iid. Then we get μ-a.s.

$$P'_\eta(0 \text{ is visited i.o.}) = \begin{cases} 0 & \text{if } E_\mu[\log^+(\eta_1)] < \infty, \\ 1 & \text{if } E_\mu[\log^+(\eta_1)] = \infty, \end{cases}$$

Thank you for your attention.