Pathwise Uniqueness for Stochastic Heat Equations with multiplicative noise: the Colored Noise Case

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Outline

Stochastic Heat Equation

Stochastic Heat Equation in $\mathbb{R}_+ \times \mathbb{R}^d$ with multiplicative noise

$$du(t,x) = \Delta u(t,x)dt + \sigma(u(t,x))W(dt\,dx), \quad t > 0, x \in \mathbb{R}^d.$$

Definition of Colored Noise

A colored noise W on $\mathbb{R}_+ \times \mathbb{R}^d$ is a signed random measure on $\mathbb{R}_+ \times \mathbb{R}^d$, s.t.

1. For $t > 0$ and bounded $A \in \mathcal{B}(\mathbb{R}^d)$ let

$$W([0,t] \times A) =: W_t(A) = \int_0^t \int_{\mathbb{R}^d} \mathbb{1}_A(x) W(ds\, dx).$$

   Then $(W_t(A))_{t \geq 0}$ is a martingale started in 0.

2. W is a Gaussian process and $\forall \phi, \psi \in C_c^\infty(\mathbb{R}^d)$:

$$\mathbb{E}[W_t(\phi) W_s(\psi)] = (t \wedge s) \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \phi(w) k(w,z) \psi(y) dw\, dz.$$

Note: $(W_t(A))_{t \geq 0}$ is a Brownian Motion with covariance $\mathbb{E}[(W_1(A))^2] = \int_A \int_A k(w,z)\, dw\, dz$.

Definition of Colored Noise

A colored noise W^k on $\mathbb{R}_+ \times \mathbb{R}^d$ is a signed random measure on $\mathbb{R}_+ \times \mathbb{R}^d$, s.t.

1. For $t > 0$ and bounded $A \in \mathcal{B}(\mathbb{R}^d)$ let

$$W^k([0,t] \times A) =: W^k_t(A) = \int_0^t \int_{\mathbb{R}^d} \mathbb{1}_A(x) W^k(ds\,dx).$$

   Then $(W^k_t(A))_{t\geq 0}$ is a martingale started in 0.

2. W^k is a Gaussian process and $\forall \phi, \psi \in C_c^\infty(\mathbb{R}^d)$:

$$\mathbb{E}[W^k_t(\phi) W^k_s(\psi)] = (t \wedge s) \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \phi(w) k(w,z) \psi(y) dw\,dz.$$

Note: $(W^k_t(A))_{t\geq 0}$ is a Brownian Motion with covariance $\mathbb{E}[(W^k_1(A))^2] = \int_A \int_A k(w,z)\,dw\,dz$.

Correlation Kernel

$$\mathbb{E}[W_t^k(\phi)W_s^k(\psi)] = t \wedge s \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \phi(x)k(x,y)\psi(y)dx\,dy.$$

The function

$$k : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}_+$$

is called the correlation kernel. Examples are

- White Noise: $k(x,y) = \delta(x-y)$ (k is degenerate here).
- Colored Noise: $|k(x,y)| \leq c_2(|x-y|^{-\alpha}+1)$, $\alpha \in (0,d)$ (Riesz-kernel).
- White Noise in d dimensions $\approx$ Colored Noise with $\alpha \approx d$.

Examples for SPDE I

$$du(t,x) = \Delta u(t,x)dt + \sigma(u(t,x))W(dt\,dx).$$

Examples:

- Let X be Super-Brownian-Motion in $d = 1$ and assume

$$X_t(dx) = u(t,x)dx.$$

Then the density $u : \mathbb{R}_+ \times \mathbb{R}^d$ solves

$$du = \Delta u\,dt + \sqrt{u}\,dW^\delta.$$

  - Δ-term: spatial movement according to heat-flow.
  - $\sqrt{u(t,x)}W(dt\,dx)$-term: branching term according to finite-variance offspring.
  - $W = W^\delta$ is white noise.

Examples for SPDE II

$$du(t,x) = \Delta u(t,x)dt + \sigma(u(t,x))W(dt\,dx).$$

More examples:

- If the branching depends on an underlying random environment on $\mathbb{R}^d$, $d \geq 1$:

$$du = \Delta u\,dt + u\,dW^k,$$

  where W^k is colored noise.

- Sandra's talk:

$$du = \Delta u\,dt + \text{ something } + \sqrt{u(1-u)}\,dW^\delta.$$

- One can do any kind of PDE with random noise.

Stochastic Heat Equation in $\mathbb{R}_+ \times \mathbb{R}^d$

$$du(t,x) = \Delta u(t,x)dt + \sigma(u(t,x))W^k(dt\,dx) \qquad (1)$$

We say the random field $u : \mathbb{R}_+ \times \mathbb{R}^d$ is a solution to the SHE started in $u(0,\cdot) = u_0(\cdot)$, if for all $\phi \in C_c^\infty(\mathbb{R}^d)$:

$$\int_{\mathbb{R}^d} \phi(x)u(t,x)dx = \int_{\mathbb{R}^d} \phi(x)u(0,x)dx + \int_0^t \int_{\mathbb{R}^d} \Delta\phi(x)u(s,x)dx\,ds + \int_0^t \int_{\mathbb{R}^d} \phi(x)\sigma(u(s,x))W(ds\,dx).$$

Sometimes abbreviate: $u_t(x) = u(t,x)$.

Questions:

- Existence (weak and strong)
- Uniqueness (in law, pathwise, strong)

Outline

1. Preparatory Steps for SPDE = Stochastic PDE
   - What is SPDE ?
   - What is Noise ?
   - Examples for SPDE
   - What is a Solution to an SPDE ?
2. Results
   - Assumptions and Weak Existence
   - Pathwise Uniqueness Results/Conjecture
3. Proofs
   - Proof for SDE
   - Proof for SPDE

Quick Answer:
In Lipschitz-case for σ use ODE-techniques to get strong existence and uniqueness.

Assumptions:

$$(H_\sigma) \quad |\sigma(u) - \sigma(v)| \leq c_1 |u - v|^\gamma, \quad u, v \geq 0.$$

$$(H_k) \quad k(x, y) \leq c_2(|x - y|^{-\alpha} + 1), x, y \in \mathbb{R}^d, \alpha > 0.$$

PROPOSITION (DALANG 99; MYTNIK, PERKINS, STURM 06)

Colored: Assume (H_k) and $\alpha \in (0, 2 \wedge d)$, σ continuous then there is a stochastically weak solution to (1).
White: If $k(x, y) = \delta(x - y)$, $d = 1$ and σ continuous then there is a stochastically weak solution to (1).

Pathwise Uniqueness I

Question:
Does pathwise uniqueness (PU) hold?
That means for u^1, u^2 two solutions for SHE with the same noise and $u_0^1 = u_0^2$ is it true that $u = u^1 - u^2 \equiv 0$ a.s.?

Barlow, Dalang, Perkins:
(PU) for SHE is an "important outstanding problem."

Theorem (Mytnik, Perkins 2009, White Noise)

Assume $d = 1$, $k(x, y) = \delta(x - y)$ and (H_σ). (PU) holds provided that

$$\frac{3}{4} < \gamma.$$

Note: Not quite what we hoped, e.g. $\sigma(u) = \sqrt{u}$: $\gamma = 1/2$.

Pathwise Uniqueness II

$$du = \Delta u dt + \sigma(u) dW^k$$

$$(H_\sigma) \quad |\sigma(u) - \sigma(v)| \leq c_1 |u - v|^\gamma, \quad u, v \geq 0.$$

$$(H_k) \quad k(x, y) \leq c_2(|x - y|^{-\alpha} + 1), x, y \in \mathbb{R}^d, \alpha > 0.$$

Conjecture, Work in Progress (Colored Noise)

Assume (H_σ), (H_k), $d \geq 1$. (PU) holds provided that

$$\alpha < 2(2\gamma - 1).$$

Remark:

- The case $\alpha = 1$ leads to the Mytnik and Perkins result.
- First ideas: Mytnik,Perkins, Sturm 2006: $\alpha < 2\gamma - 1$ implies (PU).

Outline

$$dX_t = \sigma(X_t)dB_t \quad \text{(one-dimensional SDE)}$$

Yamada-Watanabe: If σ's Hölder-exponent $\gamma \geq \frac{1}{2}$, then (PU).

Proof-sketch: Let $a_n \searrow 0$ and $\psi_n \in C_c^\infty(\mathbb{R}^d)$ s.t.

$$\operatorname{supp}(\psi_n) \subset (a_n, a_{n-1}), \quad \int_{a_n}^{a_{n-1}} \psi_n(x)dx = 1, \quad \psi_n(x) \leq \frac{2}{nx}.$$

Define

$$\rho_n(x) = \int_0^{|x|} dy \int_0^y dz\, \psi_n(z) \in C^2, \quad \rho_n(x) \nearrow |x|.$$

By Itô:

$$\rho_n(u_t^1 - u_t^2) = \text{martingale} + \frac{1}{2}\int_0^t \psi_n(|u_s^1 - u_s^2|)(\sigma(u_s^1) - \sigma(u_s^2))^2 \, ds.$$

Take expectations

$$\mathbb{E}[\rho_n(u_t)] \leq \mathbb{E}[\int_0^t \frac{2}{n|u_s|}|u_s|^{2\gamma} \, ds].$$

LHS tends to $\mathbb{E}[|u_t|]$. By $2\gamma - 1 \geq 0$ and Gronwall's Lemma get the result.

Adapt the previous proof!

- For $x \in \mathbb{R}^d$ choose a function $\Phi_x^n : \mathbb{R}^d \to \mathbb{R}_+$, s.t.

$$\operatorname{supp}(\Phi_x^n) \subset B(x, m_n).$$

- $\rho_n(\langle u_t, \Phi_x^n \rangle)$ approximates $|u_t(x)| = |u_t^1(x) - u_t^2(x)|$.

For the semi-martingale $\rho_n(\langle u_t, \Phi_x^n \rangle)$ use Itô:

$$\rho_n(\langle u_t, \Phi_x^n \rangle) = \cdots + \frac{1}{2} \int_0^t \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \psi_n(|\langle u_s, \Phi_x^n \rangle|) \Phi_x^n(w)(\sigma(u_s^1(w)) - \sigma(u_s^2(w))) \Phi_x^n(z)(\sigma(u_s^1(z)) - \sigma(u_s^2(z))) k(w, z) dw\, dz\, ds.$$

Need to estimate

$$\rho_n(\langle u_t, \Phi_x^n \rangle) = \frac{1}{2} \int_0^t \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \psi_n(|\langle u_s, \Phi_x^n \rangle|) \\ \Phi_x^n(w)(\sigma(u_s^1(w)) - \sigma(u_s^2(w))) \\ \Phi_x^n(z)(\sigma(u_s^1(z)) - \sigma(u_s^2(z))) \\ k(w,z) dw\, dz\, ds.$$

$$\leq C \int_0^t \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{2}{n a_n} \mathbb{1}\{|\langle u_s, \Phi_x^n \rangle| \leq a_{n-1}\} \\ \Phi_x^n(w)|u_s(w)|^\gamma \Phi_x^n(z)|u_s(z)|^\gamma \\ (|w-z|^{-\alpha} + 1) dw\, dz\, ds$$

$$=: I_n(x,t).$$

$$\frac{2}{na_n} \times \mathbb{1}\{|\langle u_s, \Phi_x^n \rangle| \le a_{n-1}\} \times \Phi_x^n(w)|u_s(w)|^\gamma \times \Phi_x^n(z)|u_s(z)|^\gamma$$

IDEA

- $u_s(\cdot)$ is small at a certain point close to $x \in \mathbb{R}^d$
- Transfer that to $u_s(w)$ for all w close to x!
- Use Hölder-regularity of $u_s(\cdot)$.

Let ξ be the Hölder-exponent of $u_s(\cdot)$.

- Elementary result by Sanz-Solé, Sarrá, 2002: $\xi < 1 - \frac{\alpha}{2}$.
- Mytnik, Perkins, Sturm 2006 for colored noise:
  Where u_s is small we can expect $\xi < 1 \wedge \frac{1-\frac{\alpha}{2}}{1-\gamma}$.
- Mytnik, Perkins 2009 for white noise:
  Where u_s is small we can expect $\xi < 2$
  (i.e. $\exists$ Hölder-continuous derivative).

Thus for $|u_s(x)| \leq a_n$, $w \in \mathbb{R}^1$, s.t. $|w - x| \leq m_n^{-1}$:

$$|u(s,w)| \approx |u_s(x)| + |w - x|^{\xi} \approx 2a_n$$

if $m_n \geq |w - x| \approx a_n^{1/\xi} \approx a_n^{1/2}$.
Thus: $m_n^{\alpha} \approx a_n^{\alpha/2}$.
Back to I_n:

$$\begin{aligned} I_n(x,t) &= c' a_n^{-1+2\gamma} a_n^{-\frac{\alpha}{\xi}} \\ &= c' a_n^{2\gamma - 1 - \frac{\alpha}{2}}. \end{aligned}$$

This tends to zero for $n \to \infty$, if

$$\alpha < 2(2\gamma - 1).$$

Thank you for your attention!

For Further Reading I

Robert Dalang.
Extending martingale measure stochastic integral with applications to spatially homogeneous s.p.d.e's.
Electronic Journal of Probability, 4(6), 1999.

Leonid Mytnik and Edwin Perkins.
Pathwise uniqueness for stochastic heat equations with Hölder continuous coefficients: the white noise case.
Probability Theory and Related Fields, 149, 2009.

Leonid Mytnik, Edwin Perkins, and Anja Sturm.
On pathwise uniqueness for stochastic heat equations with non-lipschitz coefficient.
Annals of Probability, 34, 2006.

- $$\frac{3}{4} = \frac{1}{2} + \frac{\alpha}{4}$$
- For $\gamma < \frac{1}{2}$ Burdzy, Mueller, Perkins, 2011 show the non-pathwise-uniqueness of the SHE: Non-uniqueness for non-negative solutions of parabolic stochastic partial differential equations.
- There is a “Hard killing model” proposed to show non-uniqueness for $\gamma < 3/4$.