Eurandom: Workshop "The expanding art of expansion"

A Central Limit Theorem for Weakly Self-Avoiding Walk in continuous space

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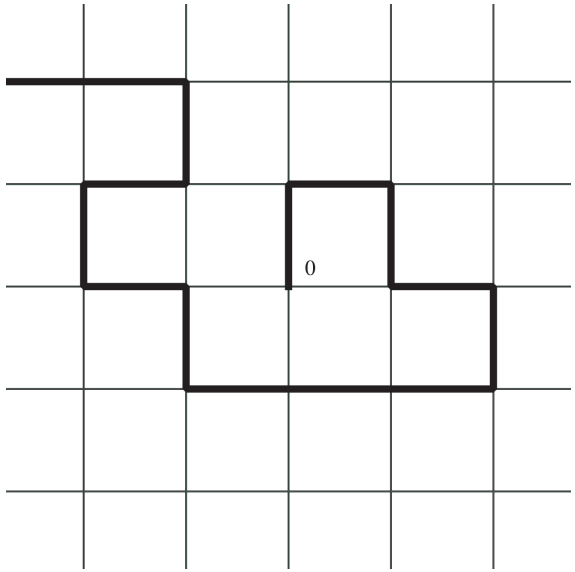
joint work with Erwin Bolthausen

Outline

1. Introduction
   - SAW and WSAW
   - A bit of literature
2. Model and result
   - WSAW on $\mathbb{R}^d$
   - LCLT
3. Sketch of the Proof
   - Convolution equation from the Lace expansion
   - A general CLT for convolution equations
4. A few comments
   - On the lace functions for this model
   - On the parameters

SAW

The *Self-Avoiding Walk* (SAW): a RW path on $\mathbb{Z}^d$ with forbidden self-intersections. Non-Markovian.

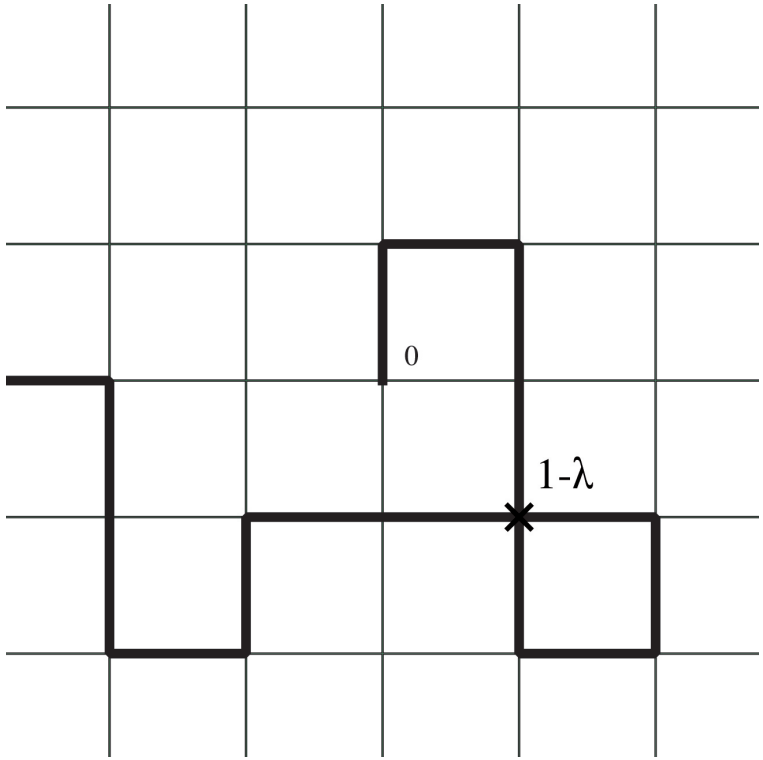


Fundamental object in statistical physics and combinatorics.
Originated in Physics and Chemistry as a model for long-chain *polymer molecules* (e.g. Proteins).

Extremely difficult to obtain rigorous results on the SAW, especially in the physically relevant low dimensions ($d = 2, 3$ and 4).

WSAW

The *Weakly Self-Avoiding Walk* (WSAW): variant of the SAW in which each self-intersection is penalized by a factor $1 - \lambda$, for some $\lambda \in (0, 1)$.



In particular, degenerate cases:

- for $\lambda = 0$ ordinary RW.
- for $\lambda = 1$ SAW.

- Brydges and Spencer (1985).
  Introduced the *Lace Expansion*: combinatorial type of expansion to study SAW and WSAW in $d \geq 5$.
  Result: WSAW is diffusive in $d \geq 5$.
  Ingredients: Lace expansion, Laplace transforms in time and inversion.

- Hara and Slade (1992).
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  A new method using Lace Expansion and a fixed point argument to prove again a local CLT for WSAW in $d \geq 5$.
  Advantages:
  - avoid Fourier spaces,
  - better error estimates,
  - quite general method.
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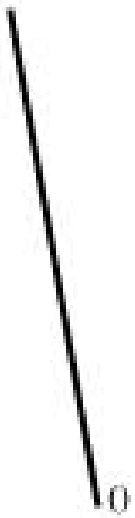

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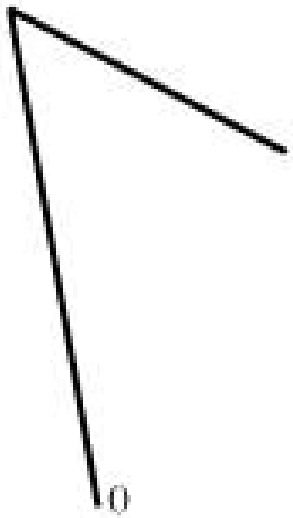

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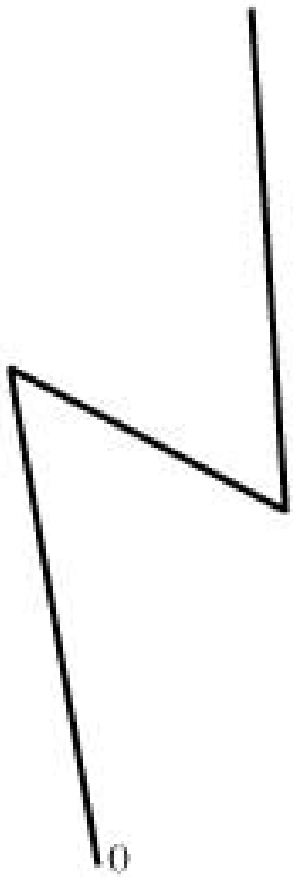

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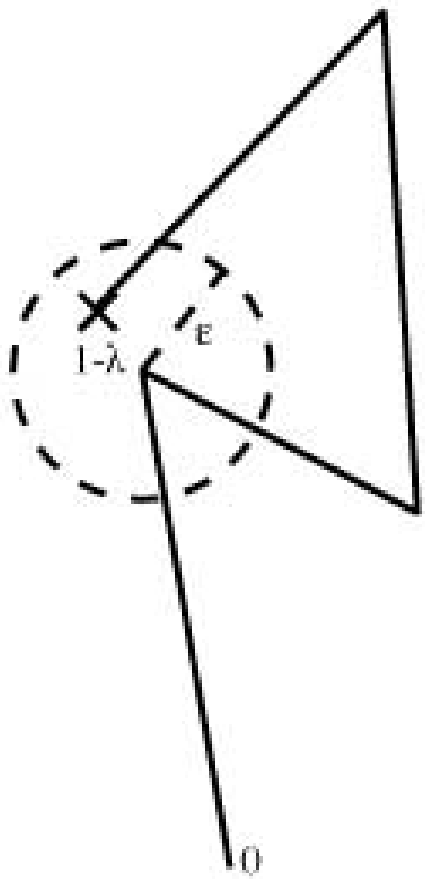

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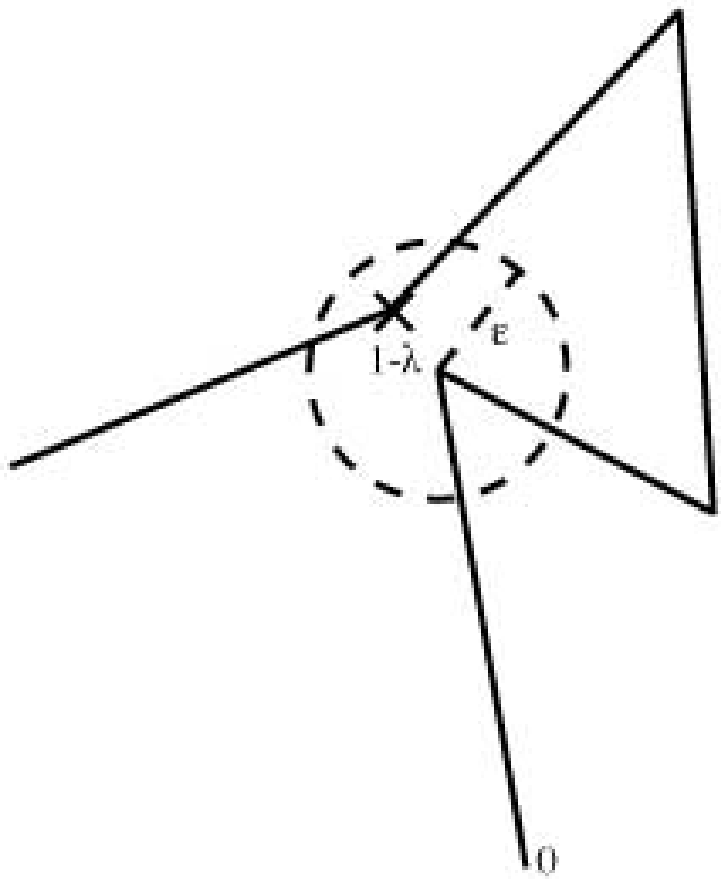

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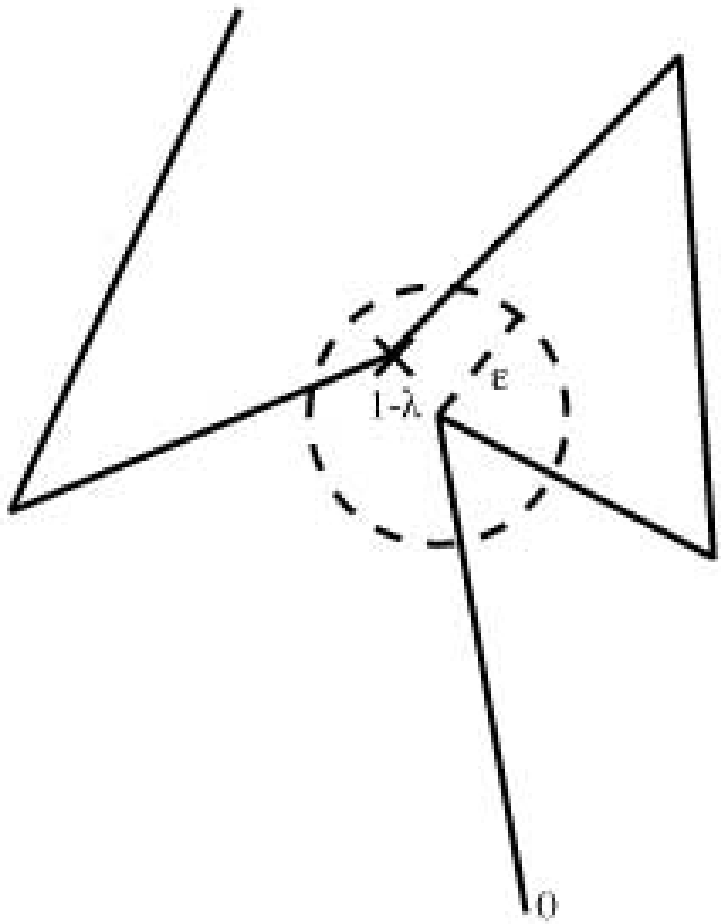

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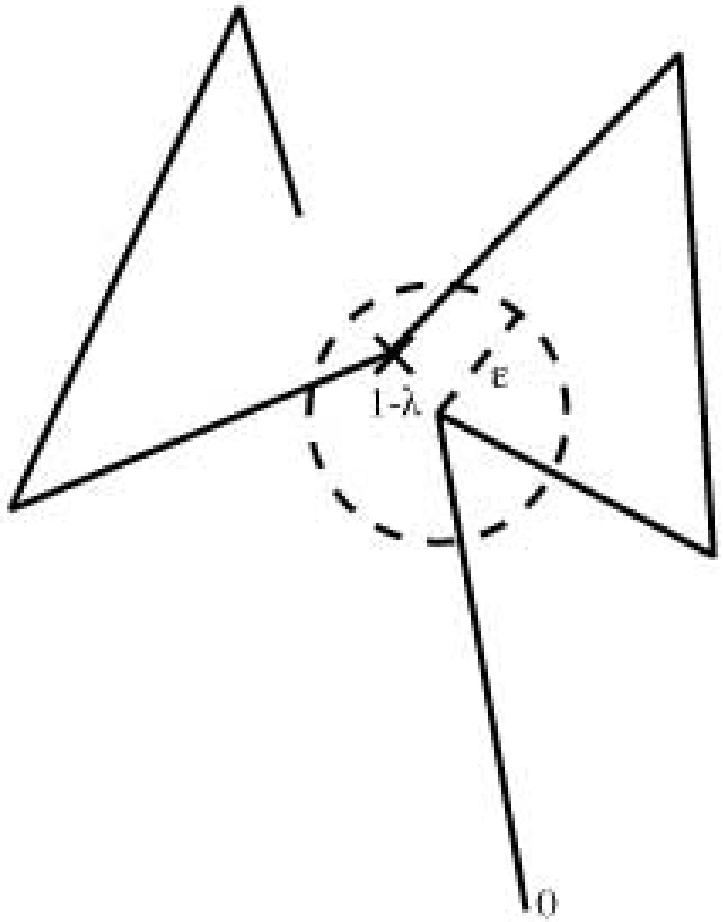

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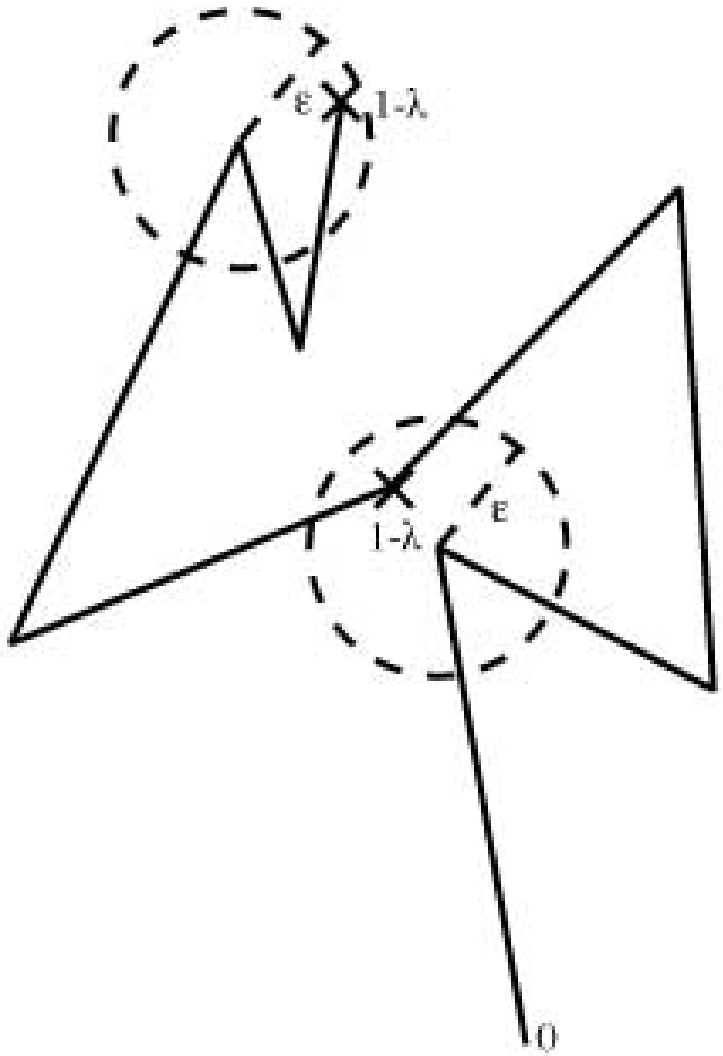

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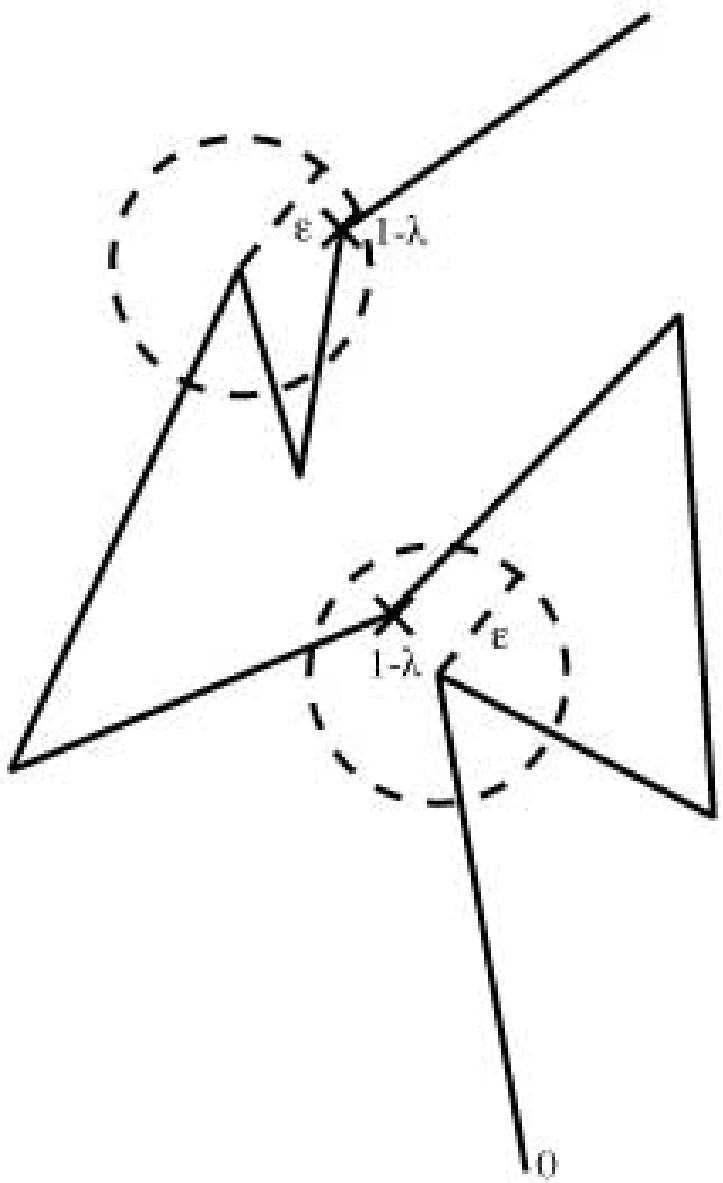

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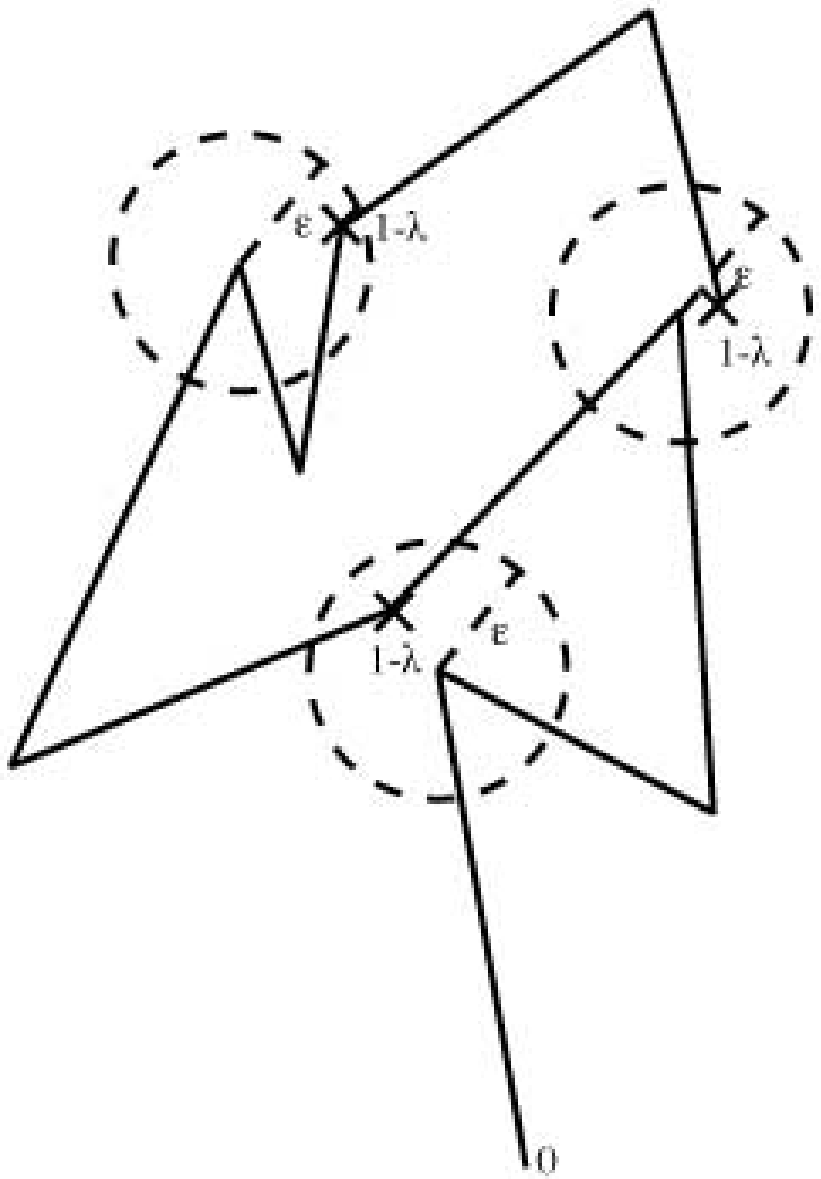

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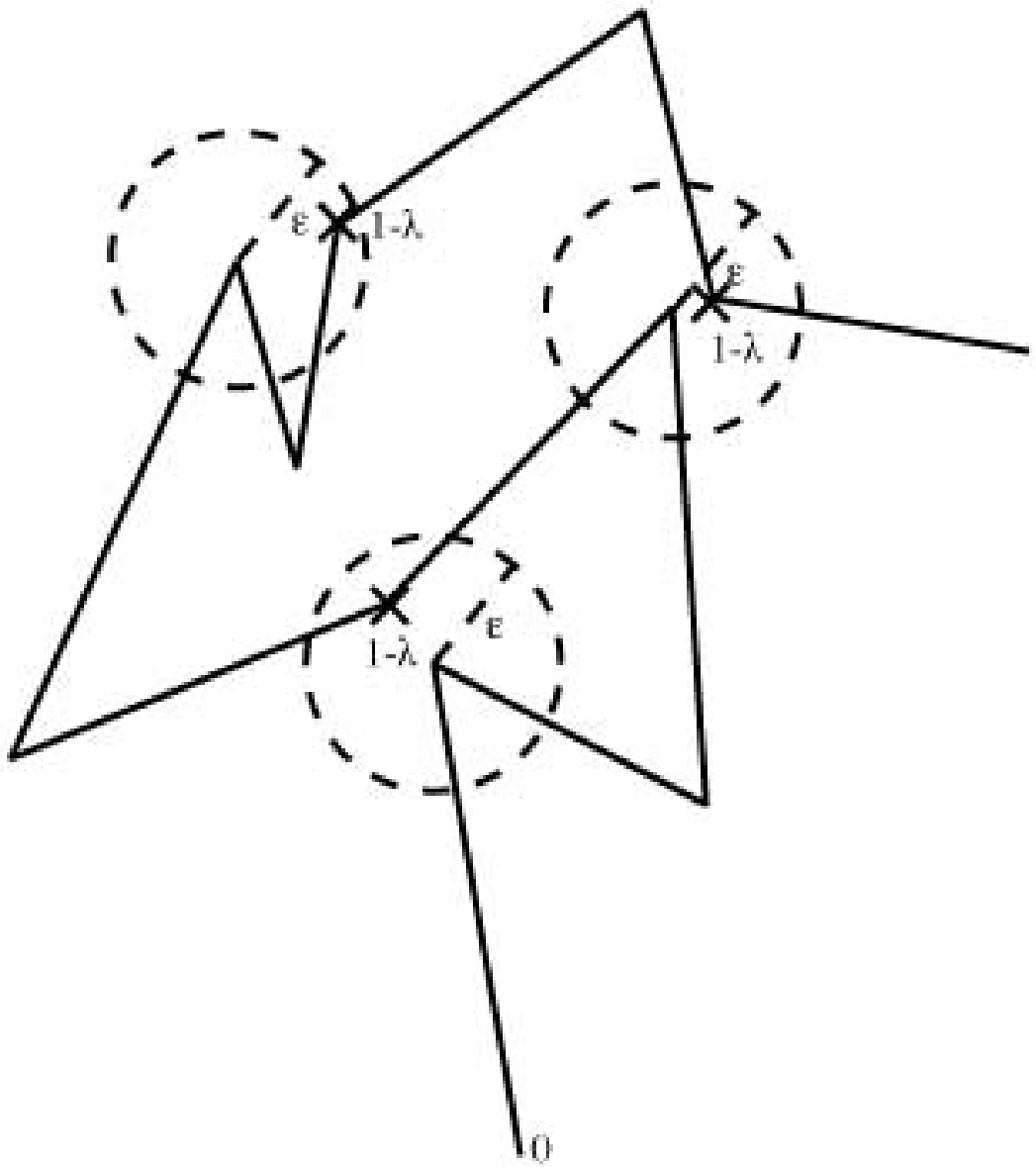

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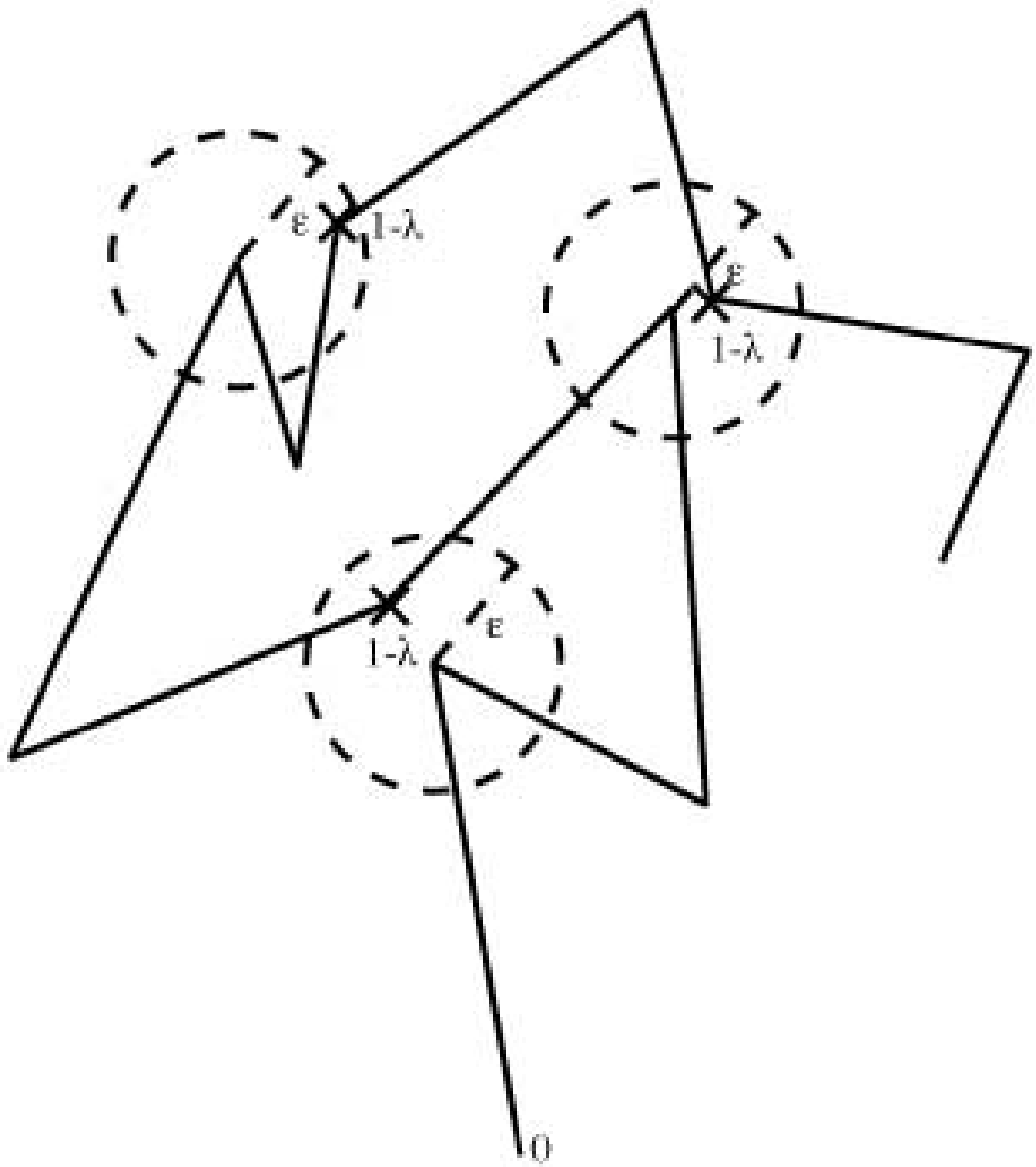

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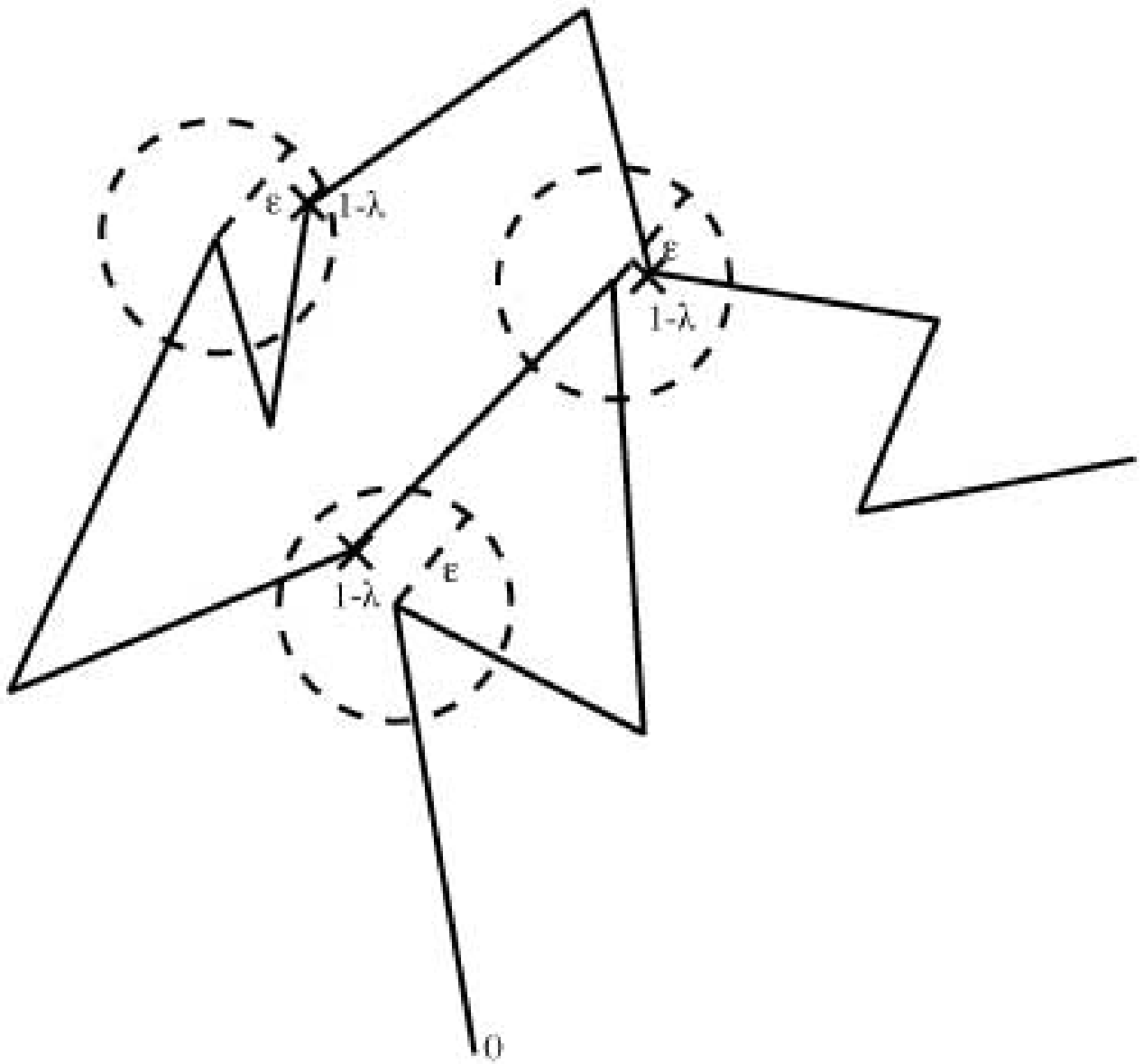

Definition

Let $\omega = (\omega_0, \omega_1, \omega_2, \ldots)$ be a path in $\mathbb{R}^d$. Fix $\epsilon > 0$. Set

$$U_{st}(\omega) := \mathbb{I}_{\{\|\omega_s - \omega_t\| \le \epsilon\}}, \quad s, t \in \mathbb{N}_0.$$

For $x \in \mathbb{R}^d$ and $n \in \mathbb{N}$. Let $D_n^0 := \{\omega \in (\mathbb{R}^d)^n : \omega_0 = 0\}$ and for $\lambda \in [0, 1]$ define the *connectivity function* as

$$C_n(x) := \int_{D_n^0} \delta_{x,\omega_n} \prod_{0 \le s < t \le n} [1 - \lambda U_{st}(\omega)] P_n^{RW}(d\omega), \quad C_0(x) := \delta_{0,x}. \quad (1)$$

Set the total masses $c_0 := 1$ and $c_n = \int_{\mathbb{R}^d} C_n(x)dx$.

$$\frac{C_n(x)dx}{c_n} = \text{ density function of the WSAW after } n \text{ steps .}$$

$P_n^{RW}(d\omega) = n$-step distribution of a RW on $\mathbb{R}^d$ with 1-step density given by a d-dim Gaussian with covariance matrix $\eta \cdot Id_d$, that is, $f(x) = (2\pi\eta)^{-\frac{d}{2}} e^{-\frac{\|x\|^2}{2\eta}}$.

Theorem

Let $d \geq 5$. Then there exists $\lambda_0 = \lambda_0(\epsilon) > 0$ such that for all $\lambda \in (0, \lambda_0)$ and $n \in \mathbb{N}$,

$$c_n = \alpha \mu^n \left(1 + O(n^{-(d-4)/2})\right). \tag{2}$$

$$\left| \frac{C_n(x)}{c_n} - \varphi_{\delta n}(x) \right| \leq K \left[n^{-1/2} \varphi_{n\nu}(x) + n^{-d/2} \sum_{j=1}^{n/2} j \varphi_{j\nu}(x) \right]. \tag{3}$$

The constants α, μ, δ and ν are positive and depend on λ, η and d, whereas K only depends on the dimension.

φ_ν stands for the d-dimensional normal density with covariance matrix $\nu \cdot Id_d$.

Convolution equation for the connectivity

From the lace expansion, we get the renewal equations for the connectivity:

$$C_n(x) = (f * C_{n-1})(x) + \sum_{m=1}^{n} (\Pi_m * C_{n-m})(x), \quad C_0(x) = \delta_{0,x}. \qquad (4)$$

and its total mass:

$$c_n = c_{n-1} + \sum_{m=1}^{n} \pi_m c_{n-m}, \quad c_0 = 1.$$

- $f(x)$= d-dimensional normal density covariance matrix $\eta \cdot Id_d$.
- Π_m= is the so called *lace function*.
- $\pi_m = \int_{\mathbb{R}^d} \Pi_m(x)dx$=total mass of the *lace function*.

Assume the exponential growth:

$$C_n(x) = \mu^n A_n(x)$$

and define $B_m := \Pi_m/(\lambda c_m)$. Denote by a_n, b_n the total masses of A_n, B_n, respectively.
Then the recursion on C_n in (4) becomes

$$A_n(x) = \mu^{-1}(f * A_{n-1})(x) + \lambda \sum_{m=1}^{n} a_m (B_m * A_{n-m})(x), \tag{5}$$

with $A_0(x) = \delta_{0,x}$, and for the total masses we have

$$a_n = \mu^{-1} a_{n-1} + \lambda \sum_{m=1}^{n} a_m b_m a_{n-m}, \quad a_0 = 1. \tag{6}$$

μ is called connectivity constant and will be determined later.

$$A_n(x) = \mu^{-1}(f * A_{n-1})(x) + \lambda \sum_{m=1}^{n} a_m (B_m * A_{n-m})(x), \quad A_0(x) = \delta_{0,x} \tag{7}$$

Theorem (General LCLT for Convolution equations)

Assume $d \geq 5$ and that $\mathbf{B} = (B_m)_{m\geq 1}$ is a sequence of functions on $\mathbb{R}^d$ satisfying

$$|B_m(x)| \leq m^{-d/2} \sum_{k=1}^{m/2} k^{1-d/2} \varphi_{k\nu}(x), \quad m \geq 1. \tag{8}$$

Then there exist $\lambda_0 > 0$ such that for all $\lambda \in (0, \lambda_0)$ the sequence $\mathbf{A} = (A_n)_{n\geq 0}$ defined in (7) satisfies for all $n \in \mathbb{N}$ and $x \in \mathbb{R}^d$:

$$|A_n(x) - a_n \varphi_{n\delta}(x)| \leq K \left[n^{-1/2} \varphi_{n\nu}(x) + n^{-d/2} \sum_{j=1}^{n/2} j\, \varphi_{j\nu}(x) \right].$$

Recall for our continuous WSAW $B_m := \Pi_m/(\lambda c_m)$.

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Proposition (Limit of the total mass a_n and connectivity constant μ)

Let $(b_m)_{m\in\mathbb{N}_0}$ *be a real-valued sequence with* $\beta := \sum_{m=1}^{\infty} m|b_m| < \infty$.

Then there is a $\lambda_0 = \lambda_0(\beta) > 0$ *such that for all* $\lambda \in (0, \lambda_0]$, *there exists a unique sequence* $(a_n)_{n\geq 0}$ *with* $a_0 = 1$, *and*

$$a_n = \mu^{-1} a_{n-1} + \lambda \sum_{m=1}^{n} a_m b_m a_{n-m}, \tag{9}$$

$$\mu^{-1} := (1 - \lambda \sum_{m=1}^{\infty} a_m b_m), \tag{10}$$

such that $\sum_n |a_n - a_{n-1}| < \infty$ *and* $a_n \in [1/C, C], C > 1$.

Proof: Define an appropriate norm and a contraction on a subspace of $(l_\infty, \|.\|_\infty)$. Apply Banach fixed point theorem. (Bolthausen, Ritzmann (2001))

Proof of the general LCLT

Recall, let $\mathbf{A} = (A_n)_{n\geq 0}$ with $A_0(x) = \delta_{0,x}$ and

$$A_n(x) = \mu^{-1}(f * A_{n-1})(x) + \lambda \sum_{m=1}^{n} a_m (B_m * A_{n-m})(x), \tag{11}$$

Claim:

$$|A_n(x) - a_n \varphi_{n\delta}(x)| \leq K \left[n^{-1/2} \varphi_{n\nu}(x) + n^{-d/2} \sum_{j=1}^{n/2} j\, \varphi_{j\nu}(x) \right] =: K \chi_n(x),$$

Strategy: consider a suitable Banach space of sequences of measures, construct a suitable contractive operator, use fixed point theorem.

A Banach space of sequences

Definition

Consider the *space of sequences*

$$\mathcal{W} = \left\{ \mathbf{G} = (G_n)_{n\geq 0} |\, G_n : \mathbb{R}^d \longrightarrow \mathbb{R} \text{ is integrable and } \|\mathbf{G}\|_\chi < \infty \right\},$$

with *norm*:

$$\|\mathbf{G}\|_\chi := \sup_x |G_0(x)| + \sup_{x,n\geq 1} \frac{|G_n(x)|}{\chi_n(x)}. \tag{12}$$

$$\chi_n(x) := n^{-1/2}\varphi_{n\nu}(x) + n^{-d/2} \sum_{j=1}^{n/2} j\, \varphi_{j\nu}(x) = \text{ the error in our LCLT.}$$

$\Rightarrow (\mathcal{W}, \|.\|_\chi)$ is a *Banach space*.

A Contractive Operator

Definition

Define the *operator* $\psi : \mathcal{W} \longrightarrow \mathcal{W}$, with $\psi(\mathbf{G})_0 = G_0$, and for $n \geq 1$

$$\psi(\mathbf{G})_n = G_n - \sum_{j=1}^{n} \varphi_{(n-j)\delta} * \left[G_j - \mu^{-1}(f * G_{j-1}) - \lambda \sum_{m=1}^{j} a_m (B_m * G_{j-m}) \right]. \quad (13)$$

Every fixed point of ψ satisfies our original recursion, namely, if

$$\psi(\mathbf{A}) = \mathbf{A}, \quad \mathbf{A} = (A_n)_{n \geq 0} \in \mathcal{W}$$

then

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Let $\mathcal{W}_0 := \{\mathbf{G} \in \mathcal{W} | G_0 \equiv 0\}$

Lemma (ψ is a contraction on $(\mathcal{W}_0, \|.\|_\chi)$)

Let $\mathbf{G} \in \mathcal{W}_0$. Then, for λ small enough, there exists $\gamma \in (0,1)$ with

$$\|\psi(\mathbf{G})\|_\chi \leq \gamma \|\mathbf{G}\|_\chi.$$

Let $\mathbf{E} := (a_n \varphi_{n\delta})_{n \geq 0}$ (our Gaussian approximations for the CLT).

Lemma ($\psi(\mathbf{E}) - \mathbf{E} \in \mathcal{W}_0$)

$$\|\psi(\mathbf{E}) - \mathbf{E}\|_\chi \leq K,$$

where K is a positive constant depending on the sequence $\mathbf{B} = (B_m)_{m \geq 1}$.

Recall $\mathbf{E} := (a_n \varphi_{n\delta})_{n\geq 0}$ with $E_0(x) := \delta_{0,x}$.

- From previous lemmas: ψ is a contraction on $(\mathcal{W}_0, \|.\|_\chi)$ and $\psi(\mathbf{E}) - \mathbf{E} \in \mathcal{W}_0$.
- By linearity of ψ and Banach fixed point theorem, there exists a unique fixed point on $\mathbf{E} + \mathcal{W}_0$, that is, a sequence $\mathbf{A} = (A_n)_{n\geq 0}$ satisfying

$$\psi(\mathbf{A})_n(x) = A_n(x), \qquad \text{for all } n, x \tag{15}$$

and

$$A_0(x) = E_0(x) := \delta_{0,x}. \tag{16}$$

- Moreover

$$\|\mathbf{A} - \mathbf{E}\|_\chi \leq K \quad \text{that is} \quad |A_n(x) - a_n \varphi_{n\delta}(x)| \leq K \chi_n(x)$$

Q.E.D.

the last step follows by:

$$\begin{aligned} \|\mathbf{A}-\mathbf{E}\|_\chi &= \| \sum_{n\geq 1} \left(\psi^{(n)}(\mathbf{E}) - \psi^{(n-1)}(\mathbf{E}) \right) \|_\chi \\ &\leq \sum_{n\geq 1} \| \psi^{(n-1)} \left(\psi(\mathbf{E}) - \mathbf{E} \right) \|_\chi \\ &\leq \|\psi(\mathbf{E}) - \mathbf{E}\|_\chi \sum_{n\geq 1} \gamma^n \leq K. \end{aligned} \tag{17}$$

where we used the contractivity and that $\|\psi(\mathbf{E}) - \mathbf{E}\|_\chi \leq K$ from the lemma.

On the lace functions

Recall, so far we assumed the recursion

$$C_n(x) = (f * C_{n-1})(x) + \sum_{m=1}^{n} (\Pi_m * C_{n-m})(x), \tag{18}$$

and the estimate

$$|B_m(x)| := \left| \frac{\Pi_m(x)}{\lambda c_m} \right| \leq m^{-d/2} \sum_{k=1}^{m/2} k^{1-d/2} \varphi_{k\nu}(x). \tag{19}$$

The recursion and the control on the *lace function* Π_m follows from adaptation of standard inductive arguments in lace expansion.

The lace function is given by the expansion in power of λ:

$$\Pi_m(x) = \sum_{N=1}^{\infty} (-\lambda)^N \Pi_m^{(N)}(x),$$

where the coefficients are the functions

$$\Pi_m^{(N)}(x) := \int_{D_n^0} \delta_{\omega_n, x} \Big[\sum_{L \in \mathcal{L}^N[0,m]} \prod_{st \in L} U_{st} \prod_{s't' \in \mathcal{C}(L)} (1 - \lambda U_{s't'}) \Big] P_n^{RW}(d\omega)$$

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Estimating the lace coefficients by the connectivities for subwalkers

In standard WSAW on $\mathbb{Z}^d$

$$\Pi_m^{(N)}(x) \leq \sum_{y_i, m_i} C_{m_1}(y_1) C_{m_2}(-y_1) C_{m_3}(y_2) C_{m_4}(y_1 - y_2) C_{m_5}(y_3 - y_1) \cdots$$
$$\cdots C_{m_{2N-3}}(x - y_{N-3}) C_{m_{2N-2}}(y_{N-2} - x) C_{m_{2N-1}}(x - y_{N-2}),$$

where the sum is over $y_1, \ldots, y_{N-2} \in \mathbb{Z}^d$ and over $m_1, \ldots, m_{2N-1}$ with $\sum_i m_i = m$.

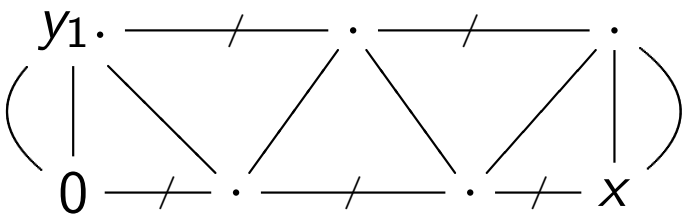

Diagram of a trajectory with N self-intersections.
Slashed lines denote subwalks that may have length zero.

Role of ϵ in our continuous setting

$$\Pi_m^{(N)}(x) \leq \sum_{m_i} \int dy_1 C_{m_1}(y_1) \int_{B_\epsilon(0)} dz_0 C_{m_2}(z_0 - y_1) \int dy_1 C_{m_3}(y_2 - z_0)$$
$$\times \int_{B_\epsilon(y_1)} dz_z C_{m_4}(z_1 - y_2) \int dy_3 C_{m_5}(y_3 - z_1) \cdots$$
$$\cdots \int_{B_\epsilon(x)} dz C_{m_{2N-3}}(z - z_{N-3}) \int_{B_\epsilon(y_{N-2})} dz_{N-2} C_{m_{2N-2}}(z_{N-2} - z)$$
$$\times C_{m_{2N-1}}(x - z_{N-2}),$$

where the sum is again over $m_1, \ldots, m_{2N-1}$ with $\sum_i m_i = m$.

ϵ-dependence of λ

Estimating inductively:

$$\Pi_m^{(N)}(x) \leq K_1 (L\epsilon^{2d})^{N-1} \mu^m \Psi_m(x), \tag{20}$$

with K_1, L positive constants and

$$\Psi_m(x) := m^{-d/2} \sum_{k=1}^{m/2} k^{1-d/2} \varphi_{k\nu}(x).$$

Finally, by (20), we get the desired bound:

$$|\Pi_m(x)| := |\sum_{N=1}^{\infty} (-\lambda)^N \Pi_m^{(N)}(x)| \leq \lambda K \mu^m \Psi_m(x)$$

if $\lambda = \lambda(\epsilon)$ is small enough.

diffusivity constant δ

Recall $\mathbf{E} := (a_n \varphi_{n\delta})_{n \geq 0}$ is our Gaussian approximating sequence for the CLT.
An expression for the variance δ comes out when proving that $\|\psi(\mathbf{E}) - \mathbf{E}\|_\chi \leq K$, and it turns out that

$$\delta = \frac{\eta \mu^{-1} + \lambda \tau}{1 + \lambda \sigma},$$

- $\eta =$ variance of the original 1-step Gaussian distribution,
- $\mu =$ connectivity constant,
- σ and τ are constants expressed in terms of the sequences $\mathbf{a}$ and $\mathbf{B}$.

THANK YOU!

Zürich