The growth of the infinite long-range percolation cluster and an application to spatial epidemics

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- One can adapt the definition of R_0 such that it has useful properties in "household epidemics" and epidemics on random graphs (configuration model/ Poissonian graphs)
- Important property: Exponential asymptotic growth of the number of infectious individuals per "infection generation"

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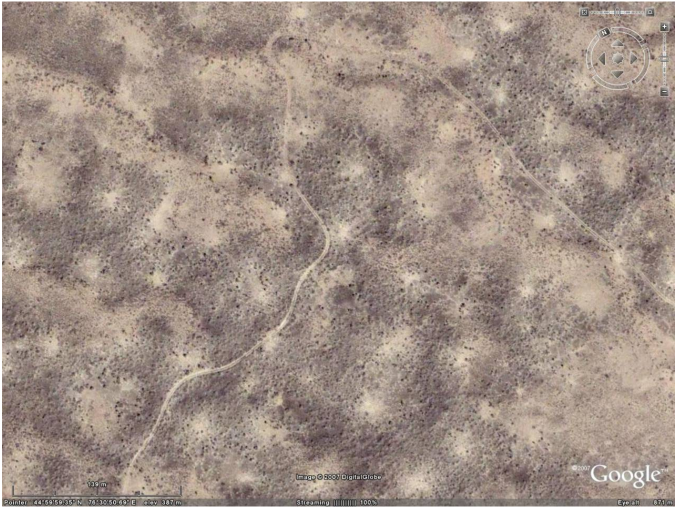
- **Example: Plague in Kazakhstan**
    - Up to several decades ago, sylvatic plague was a considerable killer in Soviet Union
    - It has been known since 1898 that plague is transmitted by fleas
    - In 1927 the cause of endemicity of plague has been identified:

Great gerbil
(Rhombomys opimus)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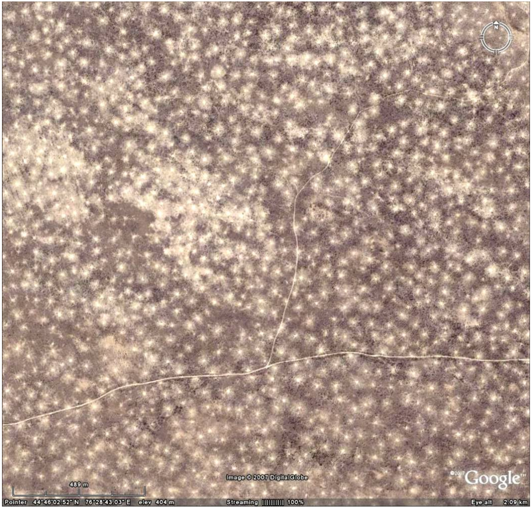
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Question: Is it possible to observe exponential growth in a spatial epidemic by allowing for long-range contacts, of which the frequency only depends on the distance?

Model: Long-range percolation

Construct a random graph with vertex set $\mathbb{Z}^d$:

- an undirected edge between vertices x and y is present with probability $p(x,y) = p(\|x-y\|) = 1 - e^{-\lambda(\|x-y\|)}$
- The presence or absence of an edge is independent of the presence or absence of other edges
- Assume that $\lambda(r)$ is non-increasing and regularly varying, i.e., $\lambda(r) = r^{-\beta}L(r)$, where $L(r)$ is slowly varying $\left(\lim_{r\to\infty} \frac{L(ar)}{L(r)} = 1 \text{ for all } a > 0\right)$
- To avoid annoying technicalities, assume
  1. $p(1) = 1$ and $p(r) < 1$ for $r > 1$
  2. the slowly varying function $L(r)$ is decreasing

Definitions

- $\mathcal{B}_k$: set of vertices within graph distance k of the origin
- $D(x,y)$: graph distance between vertices x and y
- $$\underline{R_0} := \liminf_{k\to\infty} (\mathbb{E}(|\mathcal{B}_k|))^{1/k}$$
- $$\overline{R_0} := \limsup_{k\to\infty} (\mathbb{E}(|\mathcal{B}_k|))^{1/k}$$

Already known

- **Berger**: If $\beta > 2d$ and $p(1) = 1$ then with probability 1,

$$\liminf_{\|x\|\to\infty} \frac{D(0,x)}{\|x\|} > 0$$

So, $D(0,x)$ grows at least linearly

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- **Biskup**: If $d < \beta < 2d$ then for all $\epsilon > 0$ and $\Delta := \frac{\log[2]}{\log[2d/\beta]} > 1$,

$$\lim_{\|x\|\to\infty} \mathbb{P}\left(\Delta - \epsilon \leq \frac{\log[D(0,x)]}{\log[\log[\|x\|]]} \leq \Delta + \epsilon\right) = 1$$

  So, $D(0,x)$ is roughly $(\log[\|x\|])^{\Delta}$

Theorem (Non-summable $p(r)$)

Let $\lambda(r) = r^{-\beta}L(r)$, where $L(r)$ is slowly varying. If either

- $\beta < d$, *or*
- $\beta = d$ *and* $\int_2^\infty L(r)r^{-1}dr = \infty$

then $\mathbb{P}(|\mathcal{B}_1| = \infty) = 1$. *In particular,* $|\mathcal{B}_k|^{1/k} = \infty$ *a.s., for* $k \in \mathbb{N}$ *and* $R_0 = \infty$.

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Proof.

The conditions on $\lambda(r)$ imply that $\sum_{x \in \mathbb{Z}^d} p(0, x) = \infty$.
By Borel-Cantelli, the origin a.s. shares an edge with infinitely many vertices, or $\mathbb{P}(\mathcal{B}_1 = \infty) = 1$ □

$d < \beta < 2d$: Non exponential growth

Theorem

Let $\lambda(r) = r^{-\beta}L(r)$, where $L(r)$ is slowly varying. If $\beta > d$, then for $k \to \infty$, $|\mathcal{B}_k|^{1/k} \to 1$ a.s. Furthermore, $R_0 = 1$

Remark: Biskup already proved (implicitly) that there is a $\delta < 1$, such that:

$$\lim_{\|x\| \to \infty} \mathbb{P}\Big(exp[D(0,x)^{\delta}] \geq \|x\| \Big) = 1.$$

In words: the probability that a given vertex, not within distance $exp[k^{\delta}]$ of the origin, is within graph distance k of the origin, vanishes if k goes to ∞.

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However, there are many vertices outside this ball of radius $exp[k^{\delta}]$, and Biskup does not provide a bound for the number of vertices at graph distance at most k of the origin.

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After that we prove the stronger theorem:

Lemma

Let $\lambda(r) = \alpha r^{-\beta}$. and let β' be such that $d < \beta' < \beta < 2d$. Then there exists a positive constant c such that for $\gamma := \frac{\log(2d/\beta')}{\log 2} < 1$, $K(n) := \exp[cn^\gamma] + 1$, all $n \in \mathbb{N}$ and all $x \in \{x \in \mathbb{Z}^d; \|x\| > K(n)\}$, it holds that

$$\mathbb{P}(D(0,x) \leq n) \leq [K(n)]^{\beta'} \|x\|^{-\beta'}.$$

Proof of lemma

If $D(0, x) \leq n$, then there is a path of at most n edges from 0 to x that contains at least one edge of length $\|x\|/n$.
Let $N(n, x; j)$ be the number of edges that are

- at the j-th position of a path of at most n edges, from 0 to x
- the first edge in the path that is longer than $\|x\|/n$

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Observe:

- By Markov's inequality: $\mathbb{P}(D(0,x) \le n) \le \sum_{j=1}^{n} \mathbb{E}(N(n,x;j))$
- $\mathbb{E}(|\mathcal{B}_n|) = \sum_{x \in \mathbb{Z}^d} \mathbb{E}(\mathbb{1}[D(0,x) \le n]) = \sum_{x \in \mathbb{Z}^d} \mathbb{P}(D(0,x) \le n)$

$d < \beta < 2d$

$$
\begin{aligned}
& \mathbb{E}(N(k,x;j)) \\
\leq & \sum_{\|x_1 - x_2\| \geq \|x\|/k} \mathbb{P}(D(0,x_1) = j-1) p(x_1,x_2) \mathbb{P}(D(x_2,x) \leq k-j) \\
\leq & \sum_{x_1,x_2 \in \mathbb{Z}^d} \mathbb{P}(D(0,x_1) = j-1) p(\|x\|/k) \mathbb{P}(D(x_2,x) \leq k-j) \\
= & \sum_{x_1 \in \mathbb{Z}^d} \mathbb{P}(D(0,x_1) = j-1) p(\|x\|/k) \sum_{x_2 \in \mathbb{Z}^d} \mathbb{P}(D(x_2,x) \leq k-j) \\
\leq & \mathbb{E}(|\mathcal{B}_{j-1}|) \mathbb{E}(|\mathcal{B}_{k-j}|) \alpha(\|x\|/k)^{-\beta},
\end{aligned}
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= & \sum_{x_1\in\mathbb{Z}^d} \mathbb{P}(D(0,x_1)=j-1)p(\|x\|/k)\sum_{x_2\in\mathbb{Z}^d} \mathbb{P}(D(x_2,x)\leq k-j) \\
\leq & \mathbb{E}(|\mathcal{B}_{j-1}|)\mathbb{E}(|\mathcal{B}_{k-j}|)\alpha(\|x\|/k)^{-\beta},
\end{aligned}
$$

Using this inequality, the lemma follows by induction.

The interesting regime

Recall $\lambda(r) = r^{-\beta} L(r)$, with $L(r)$ slowly varying

Theorem

- $\beta = d$
- $\int_2^\infty \frac{L(r)}{r} dr < \infty$ *(summability)*
- $-\int_2^\infty \frac{\log[L(r)]}{r(\log[r])^2} dx < \infty$

then there exist constants $1 < a_1 < a_2 < \infty$*, such that*

$$\lim_{k\to\infty} \mathbb{P}(a_1 < |\mathcal{B}_k|^{1/k} < a_2) = 1.$$

Furthermore, $1 < \underline{R_0} \leq \overline{R_0} < \infty$*.*

Remark

The condition $-\int_2^\infty \frac{\log[L(r)]}{r(\log[r])^2} dx < \infty$ is used in the proof, it is not known whether is necessary for the theorem to hold.

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$$\lambda(r) = \frac{1}{r^d (log[r])^\alpha}$$

satisfies the conditions for $\alpha > 1$

Proofs of $\lim_{k\to\infty} \mathbb{P}(|\mathcal{B}_k|^{1/k} < a_2) = 1$ and $\overline{R_0} < \infty$ immediately follow by stochastic domination by a branching process with offspring distribution as $|\mathcal{B}_1|$

Idea of proofs of $\lim_{k\to\infty} \mathbb{P}(|\mathcal{B}_k|^{1/k} > a_1) = 1$ and $\underline{R}_* > 1$

Recall that we assume that $p(1) = 1$. So, all vertices of $\mathbb{Z}^d$ are in the infinite component.
Furthermore, we assume that $L(x)$ is decreasing.

Construct a hierarchy of blocks with sides $l_i = (l_0)^{2^i}$ on $\mathbb{Z}^d$, such that

1. the blocks at the same level form a partition of $\mathbb{Z}^d$
2. a block at level i is entirely contained in a block at level $i+1$

Let $\mathcal{D}_i(x)$ be the set of vertices within the same level i block as x, that are connected to x by a path of length at most

$$h_i := (l_0^d + 1)2^i - 1,$$

where all vertices in the path are in this i-block.
Note: $h_{i+1} = 2h_i + 1$.

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Note: $h_{i+1} = 2h_i + 1$.
Let $m_0 = (l_0)^d$ and $m_{i+1} = c_0 L(l_{i+1}) \prod_{j=0}^{i} m_i$,
for some given constant c_0. This implies

$$m_{i+1} = L(l_{i+1})(c_0(l_0)^d \prod_{j=1}^{i} [L(l_j)]^{2^{-j}})^{2^i}$$

- Definition: A vertex is always *good* up to level 0
- Vertex x is good up to level $i+1$, if it is good up to level i, and $\mathcal{D}_i(x)$ contains vertices that share an edge with at least m_{i+1} vertices that are
  1. all good up to level i
  2. all in different level i blocks
  3. all in the same level $i+1$ block as x
- It is straightforward to show that if x is good up to level i, $\mathcal{D}_i(x)$ contains at least $\prod_{j=0}^{i} m_j$ vertices all within graph distance h_i of each other

If we choose l_0 large enough, then

- Conditioned on x being good up to level i, it is also good up to level $i+1$ with probability at least $1 - 4^{-2^i}$.
(By induction and assumption $-\int_2^\infty \frac{\log[L(r)]}{r(\log[r])^2} dx < \infty$)
- This implies that x is good up to all levels (ultimately good) with probability at least
$1 - \sum_{j=0}^{i-1} 4^{-2^j} \geq 1 - \sum_{j=0}^{i-1} 4^{-(j+1)} = 1 - \frac{1-4^{-i}}{3} \geq 2/3$
- If x is ultimately good, then at least $\prod_{j=0}^{i} m_j$ vertices are within graph distance k for $h_i \leq k < h_{i+1}$

- It is straightforward to prove that there exists $a_1 > 1$ such that $\inf_{i \in \mathbb{N}} (\prod_{j=0}^{i} m_j)^{(h_{i+1})^{-1}} > a_1$
- After some tricks one can use Kolmogorov's zero-one law to show that the number of ultimately good vertices is infinite

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- After some tricks one can use Kolmogorov's zero-one law to show that the number of ultimately good vertices is infinite
- Because $p(1) = 1$, the origin is almost surely connected to an ultimately good vertex in a finite number (say Y) of steps. Which implies that

$$\begin{aligned}\liminf_{k\to\infty} |\mathcal{B}_k|^{1/k} &= \liminf_{k\to\infty} |\mathcal{B}_{k+Y}|^{1/(k+Y)} \\ &\geq \liminf_{i\to\infty} (\prod_{j=0}^{i} m_j)^{(h_{i+1}+Y)^{-1}} \\ &> a_1\end{aligned}$$

Some open problems

- does R_0 exists in the regime where $1 < \underline{R_0} \leq \overline{R}_0 < \infty$?
  or: does $(\mathcal{B}_k)^{1/k}$ converge in some sense?
- is it possible to deduce R_0 from $\lambda(r)$?
- For $1 < \underline{R_0} \leq \overline{R}_0 < \infty$, does the long-range percolation graph contain a non-amenable subgraph?
  (A graph V is non-amenable if $\inf_{W \subset V, |W| < \infty} \frac{|\delta W|}{|W|} > 0$, where δW is the set of vertices in W with at least 1 neighbor in $V \setminus W$)

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