Asymptotic expected number of passages of a random walk through an interval

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joint work with

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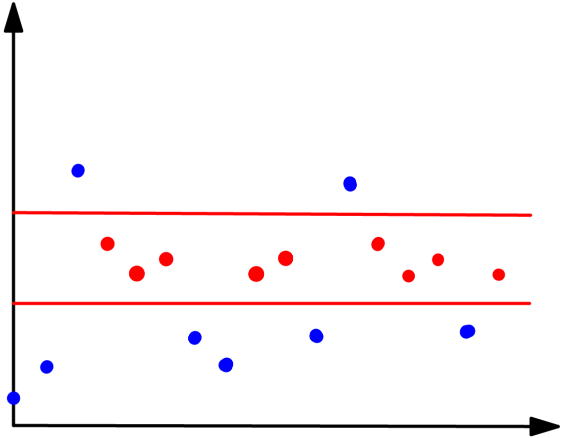
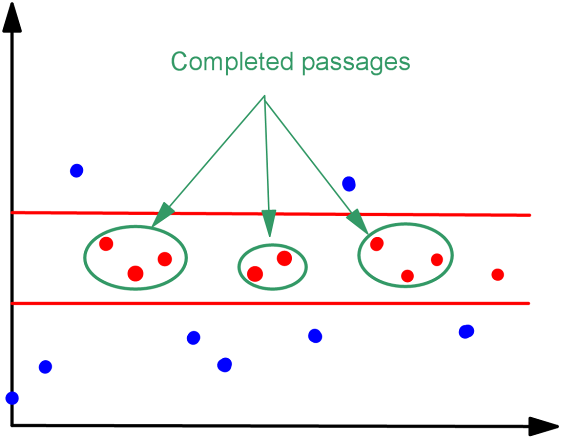
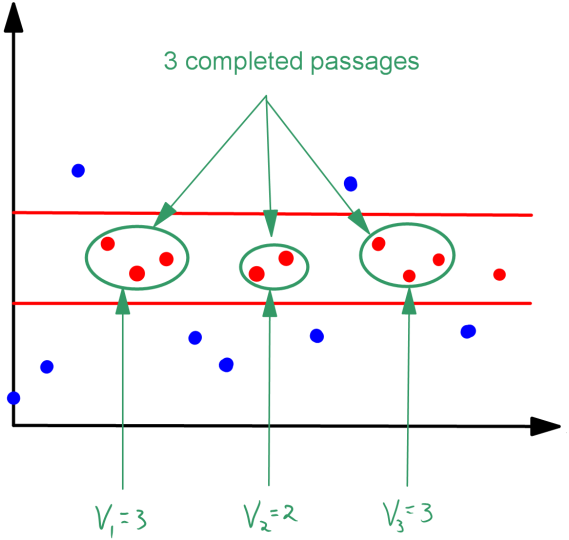
- V_i - length of the ith completed passage (if exists).




Completed passages


3 completed passages
V1=3
V2=2
V3=3

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If yes, then what is it?

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- Hence, $L_k/k \to EI_0(1 - I_1) = p_{10} = p_{01} > 0$.
- In particular, $L_k \to \infty$.

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- Now,

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- $$\sum_{i=1}^{L_{N_n}} V_i \le \sum_{i=1}^{n} V_i < \sum_{i=1}^{L_{N_n+1}} V_i$$

Hence, from

$$\frac{L_{N_n}}{n}\frac{1}{L_{N_n}}\sum_{i=1}^{L_{N_n}} V_i \leq \frac{1}{n}\sum_{i=1}^{n} V_i < \frac{L_{N_n+1}}{n}\frac{1}{L_{N_n+1}}\sum_{i=1}^{L_{N_n+1}} V_i$$

we have that

$$\frac{1}{n}\sum_{i=1}^{n} V_i \to v \quad \Longleftrightarrow \quad \frac{1}{L_k}\sum_{i=1}^{L_k} V_i \to v$$

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and so

$$\frac{1}{L_k} \sum_{i=1}^{L_k} V_i = \frac{\frac{1}{k} \sum_{i=1}^{L_k} V_i}{\frac{L_k}{k}} \to \frac{p_1}{p_{10}}$$

Proposition:
$\{S_n\}$ is stationary and ergodic, and A is such that $P(S_0 \in A, S_1 \notin A) > 0$, then, a.s.

$$\frac{1}{n}\sum_{i=1}^{n} V_i \to \frac{1}{P(S_1 \notin A | S_0 \in A)}$$

In fact, stationary and ergodic is sufficient, but all that we really used was only that

$$\frac{1}{k}\sum_{i=0}^{k} 1_{\{S_i \in A\}} \geq \frac{1}{k}\sum_{i=0}^{k} 1_{\{S_0 \in A, S_{i+1} \notin A\}}$$

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- $P(\text{ever reaching } i - 1 | \text{start from } i) = q/p = r.$

Gambler's ruin:

$$P(\text{reach } N \text{ before } 0 | \text{start from } i) = \frac{1 - r^i}{1 - r^N}$$

$$P(\text{reach } 0 \text{ before } N | \text{start from } i) = \frac{r^i - r^N}{1 - r^N}$$

In particular:

$$P(\text{reach } h+1 \text{ before } 0|\text{start from } 1) = \frac{1-r}{1-r^{h+1}}$$
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For the case $h = \infty$, clearly $EN^x = 1$ for all x hence also in the limit.

Theorem: Let $0 < h < \infty$,

(a) If X has a nonarithmetic distribution,

$$\lim_{x\to\infty} EN^x = \frac{E|X| \wedge h}{EX} \ .$$

(b) If X has an arithmetic distribution with span α then the same holds for $h \in \{k\alpha|\ k \geq 0\}$.

Theorem: Let $h = \infty$ and $EX^+ < \infty$. Then

$$\lim_{x \to \infty} EN^x = \frac{EX^+}{EX} \ .$$

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$$\frac{EX^{+}}{EX} = \frac{EX}{EX} = 1$$

as obtained earlier.

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- $EX^+ = p$.

Thus

$$\frac{E|X| \wedge h}{EX} = \frac{1}{p-q}$$

For the random walk case:

- $\alpha = 1$.
- $EX = p - q$.
- $|X| = 1 \Rightarrow |X| \wedge h = 1 \Rightarrow E|X| \wedge h = 1$
- $EX^+ = p$.

Thus

$$\frac{E|X| \wedge h}{EX} = \frac{1}{p-q}$$

$$\frac{EX^+}{EX} = \frac{p}{p-q}$$

as obtained earlier.

In fact, whenever $P(|X| \leq b) = 1$ and $h \geq b$, then

$$\frac{E|X| \wedge h}{EX} = \frac{E|X|}{EX}$$

independent of h.

THAT'S IT FOR NOW...

... or is there time to show the proof?

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- $R((-\infty, x]) \to 0$ as $x \to -\infty$.
- $R((x, x+h]) \leq ah + b$ for all x, h.

$$EN^x = E\sum_{n=0}^{\infty} 1_{\{S_n \notin (x,x+h], S_{n+1} \in (x,x+h]\}}$$

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\begin{aligned}
EN^x &= E\sum_{n=0}^{\infty} 1_{\{S_n \notin (x,x+h], S_{n+1} \in (x,x+h]\}} \\
&= \sum_{n=0}^{\infty} P(S_n \in (-\infty, x] \cup (x+h, \infty), S_{n+1} \in (x, x+h]) \\
&= \sum_{n=0}^{\infty} P(S_n \in ((-\infty, x] \cup (x+h, \infty)) \cap (x-X, x+h-X])
\end{aligned}
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&= \sum_{n=0}^{\infty} P(S_n \in ((-\infty, x] \cup (x+h, \infty)) \cap (x-X, x+h-X]) \\
&= ER((x-X^+, x-X^+ + X^+ \wedge h]) \\
&\quad +ER((x+h+X^- - X^- \wedge h, x+h+X^-])
\end{aligned}
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&= ER((x - X^+, x - X^+ + X^+ \wedge h]) \\
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&\to \frac{EX^+ \wedge h}{EX} + \frac{EX^- \wedge h}{EX}
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&\to \frac{EX^+ \wedge h}{EX} + \frac{EX^- \wedge h}{EX} = \frac{E|X| \wedge h}{EX}
\end{aligned}
$$

Note:

$$
R((y, y + X^{\pm} \wedge h]) \leq R((y, y + h]) \leq ah + b
$$

For $h = \infty$ there is a similar argument only that in the end one has

$$ER((x - X^+, x]) \to \frac{EX^+}{EX}$$

For $h = \infty$ there is a similar argument only that in the end one has

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which is allowed since $R((x - X^+, x]) \le aX^+ + b$ and $EX^+ < \infty$.

REALY THE END!