Queueing models with matrix-exponential distributions and rational arrival processes

Analytic Methods in Queueing Systems, Eurandom 2/11 2012

Bo Friis Nielsen
Informatics and Mathematical Modelling
Technical University of Denmark

Joint work with

Nigel G Bean
School of Mathematical Sciences
University of Adelaide

Extension to QBD with RAP components

- ME/RAP expressions are analytically identical to PH/MAP expressions.
- Proof of the matrix geometric formula and other matrix analytic formulas rely on path wise arguments for countable state Markov chains.
- A queue with ME (RAP) components can not be formulated as a Markov process with countable state space.
- We have prooved the MGM formula in the QBD case by two different approaches.
  - ◇ An approach, where we follow a line of proof similar to the one in Ramaswami95.
  - ◇ An approach where we apply an operator geometric result by Tweedie82 for discrete time Markov chains with general state space.

Phase Type distribution

- (Jensen49), Neuts75
- A Phase type distribution is the distribution of the time to absorption in a Markov chain with p transient states.
- Infinitessimal generator

$$Q = \begin{pmatrix} S & \boldsymbol{s} \\ \mathbf{0} & 0 \end{pmatrix}$$

S is a sub generator.

- τ : Time to absorption.
- $J(t)$: State/phase value at t $(J(t) = p+1; t \geq \tau)$.
- $\mathbb{P}(J(0) = i) = \alpha_i$, $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_p)$, frequently $\boldsymbol{\alpha}\boldsymbol{e} = 1$, where $\boldsymbol{e}$ is a vector of ones of appropriate dimension.
- $f(x) = \boldsymbol{\alpha} e^{Sx} \boldsymbol{s} \qquad \mathbb{P}(\tau > x) = e^{Sx}\boldsymbol{e}$.

Matrix exponential distribution

- $f(x) = \boldsymbol{\beta} e^{Tx} \boldsymbol{t} \qquad \mathbb{P}(\tau > x) = \boldsymbol{\beta} e^{Tx} (-T)^{-1} \boldsymbol{t}$
- $H(s) = \mathbb{E}\left(e^{-s\tau}\right) = \frac{f_1 s^{p-1} + f_2 s^{p-2} \cdots + f_p}{s^p + g_1 s^{p-1} + g_2 s^{p-2} \cdots + g_p}$
- The span of the residual life operator is finite–dimensional
- The representation $(\boldsymbol{\beta}, T, \boldsymbol{t})$ is not unique, but $T = S$ can be chosen such that $\boldsymbol{t} = -S\boldsymbol{e} = \boldsymbol{s}$ and $(\boldsymbol{e}_i, S, -S\boldsymbol{e})$ is a representation for all i.
- $f(x) = \frac{2}{3} e^{-x}(1 + \cos(x))$ is matrix exponential but not phase type

Markovian Arrival Processes (MAP)

- Neuts79, Lucantoni et al.90
- Parameterized by two matrices (D_0, D_1), $D = D_0 + D_1$ is a generator, D_0 a sub-generator, and D_1 non-negative.
- For $D_1 = \boldsymbol{d}\boldsymbol{\theta}$ we have a phase-type renewal process
- Bivariate state space $X(t) = (N(t), J(t))$ with generator

$$Q = \begin{pmatrix} D_0 & D_1 & 0 & \dots \\ 0 & D_0 & D_1 & \dots \\ \vdots & \vdots & \vdots & \vdots \end{pmatrix}$$

- $E\left(z^{N(t)}\right) = \boldsymbol{\theta} e^{(D_0 + D_1 z)t} \boldsymbol{e} \qquad \left(e^{(-\lambda + \lambda z)t}\right)$
- Joint density
  $\boldsymbol{\theta} e^{D_0 x_1} D_1 e^{D_0 x_2} D_1 \dots e^{D_0 x_n} D_1 \boldsymbol{e} \qquad \left(e^{-\lambda x_1} \lambda \dots e^{-\lambda x_n} \lambda\right)$

Rational arrival processes

- Asmussen and Bladt99, (Mitchell01)
- A process is RAP if the measure of the prediction process varies in a finite dimensional space.
- There exist matrices D_0, D_1, a row vector $\boldsymbol{\alpha}$ and a column vector $\boldsymbol{d}$, such that

$$f(x_1, \ldots, x_n) = \boldsymbol{\alpha} e^{D_0 x_1} D_1 e^{D_0 x_2} D_1 \ldots e^{D_0 x_n} \boldsymbol{d}$$

- the parameters can be chosen such that $\boldsymbol{d} = D_1 \boldsymbol{e}$, $(D_0 + D_1)\boldsymbol{e} = \boldsymbol{0}$, the maximum eigenvalue of D_0 is negative, and the maximum eigenvalue of $D_0 + D_1$ is 0.

MAP/PH/1 queue

$$Q = \begin{pmatrix} D_0 & D_1 \otimes \boldsymbol{\alpha} & 0 & 0 & \ldots \\ I \otimes \boldsymbol{s} & D_0 \oplus S & D_1 \otimes I & 0 & \ldots \\ 0 & I \otimes \boldsymbol{s\alpha} & D_0 \oplus S & A_1 \otimes I & \ldots \\ 0 & 0 & I \otimes \boldsymbol{s\alpha} & D_0 \oplus S & \ldots \\ \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix}$$

- The RAP/ME/1 queue can not be formulated as a Markov process with countable state space

Why ME? (RAP)

- Minimal dimension representation
- Potential for unique representation
- ME includes all distributions with rational transform

Then why not ME? (RAP)

- The question of whether a pair represents a distribution (process) is not resolved

A QBD with RAP components

- Define the bivariate process $N(t), \boldsymbol{A}(t)$ such that the elementary probability of an upward jump at time t is $\boldsymbol{A}(t)A_0\boldsymbol{e}\mathrm{d}t$ and the elementary probability of a downward jump is $\boldsymbol{A}(t)A_2\boldsymbol{e}\mathrm{d}t$.

- or equivalently $e^{A_1 t}A_2\boldsymbol{e}$ and $e^{A_1 t}A_0\boldsymbol{e}$ are degenerate competing ME densities.

- The value of $\boldsymbol{A}(t)$ after an upward jump is $\boldsymbol{A}(t-)A_0/\boldsymbol{A}(t)A_0\boldsymbol{e}$, the value of $\boldsymbol{A}(t)$ after a downward jump is $\boldsymbol{A}(t-)A_2/\boldsymbol{A}(t)A_2\boldsymbol{e}$.

- Between jumps $\boldsymbol{A}(t)$ evolves deterministically due to the equation

$$\boldsymbol{a}'(t) = \boldsymbol{a}(t)A_1(I - \boldsymbol{e}\boldsymbol{a}(t))$$

- The matrix Q represents the process

$$Q = \begin{pmatrix} B_0 & A_0 & 0 & 0 & \dots \\ B_1 & A_1 & A_0 & 0 & \dots \\ 0 & A_2 & A_1 & A_0 & \dots \\ 0 & 0 & A_2 & A_1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix}$$

- Such a process obviously exists
  - ◇ The MAP/PH/1 queue is a trivial example
  - ◇ The same matrix form would apply for a RAP/ME/1 queue
- The problem of determining when a given Q is a QBD-RAP matrix is harder.

Censored process

Let $\{\boldsymbol{B}(t)\}_{t\geq 0}$ be the phase vector of the censored process consisting of level m only, measured in the local time of level m, and with level $m-1$ taboo.

Theorem 1 *The total lifetime* $\ell_m(\infty)$ *of* $\{\boldsymbol{B}(t)\}_{t\geq 0}$ *is ME distributed, that is*

$$\mathbb{P}(\ell_m(\infty) > t | \boldsymbol{B}(0) = \boldsymbol{a}) = \boldsymbol{a} e^{Ut} \boldsymbol{e},$$

for some matrix U.

Expected value of phase at return to lower levels

The distribution of the return state

$$\psi(\mathcal{B}; \boldsymbol{a}) = \mathbb{P}(\boldsymbol{A}(\tau_{n-1}) \in \mathcal{B}, \tau_{n-1} < \infty | X(0) = (n, \boldsymbol{a})), \qquad \mathcal{B} \subset \mathcal{A},$$

The expected return state

$$\Psi(\boldsymbol{a}) = \mathbb{E}\left[\boldsymbol{A}(\tau_{n-1}) I(\tau_{n-1} < \infty) | X(0) = (n, \boldsymbol{a})\right] = \int_{\mathcal{A}} \boldsymbol{b} \psi(\mathrm{d}\boldsymbol{b}; \boldsymbol{a}).$$

Expected return from restricted path

$$\Psi_k(\boldsymbol{a}) = \mathbb{E}\left[\boldsymbol{A}(\tau_{n-1}) I(\tau_{n-1} < \tau_{n+k}) | X(0) = (n, \boldsymbol{a})\right].$$

The matrix G

Lemma 2 *For $k \geq 1$, the vector valued functions $\Psi_k(\boldsymbol{a})$ are linear, that is, for all $\boldsymbol{a} \in \mathcal{A}$, $\Psi_k(\boldsymbol{a}) = \boldsymbol{a}G_k$, for a unique matrix G_k. Further, $\boldsymbol{a}G_k \in \mathcal{A}$, for all $\boldsymbol{a} \in \mathcal{A}$.*

Theorem 3 *For all $\boldsymbol{a} \in \mathcal{A}$, we have*

$$\Psi(\boldsymbol{a}) = \mathbb{E}\left[\boldsymbol{A}(\tau_{n-1}) I(\tau_{n-1} < \infty) | X(0) = (n, \boldsymbol{a})\right] = \boldsymbol{a}G,$$

for a unique matrix G. Further, $\boldsymbol{a}G \in \mathcal{A}$, for all $\boldsymbol{a} \in \mathcal{A}$.

With this we can prove Theorem 1 on the total lifetime $\ell_m(\infty)$.

The matrix geometric solution - R

Theorem 4 *Assume that $X(\cdot)$ is an ergodic Markov process.*

1. *Let the vectors $\boldsymbol{\pi}_n, n \geq 0$, denote* $\lim_{t\to\infty} \mathbb{E}\left[\boldsymbol{A}(t) I(L(t) = n) | X(0) = (j, \boldsymbol{a})\right]$, *then*

$$\boldsymbol{\pi}_{n+1} = \boldsymbol{\pi}_n R \qquad \textit{for all } n \geq 1,$$

with

$$R = A_0(-U)^{-1}.$$

2. *The vectors $\boldsymbol{\pi}_0$ and $\boldsymbol{\pi}_1$ satisfy*

$$\boldsymbol{\pi}_1 \left(A_1 + B_2(-B_1)^{-1} B_0 + R A_2\right) = 0, \quad \boldsymbol{\pi}_0 = \boldsymbol{\pi}_1 B_2(-B_1)^{-1},$$

subject to

$$\boldsymbol{\pi}_1 \left(B_2(-B_1)^{-1}\boldsymbol{e} + (I - R)^{-1} \boldsymbol{e}\right) = 1.$$

A discrete time Markov chain on a general state space - Tweedie82

- A discrete time Markov $X_n = (N_n, A_n)$ chain on the state space $\mathbb{N} \times \mathcal{E}$
- Consider the kernel (here in QBD-version)

$$\tilde{P}(x, \mathcal{B}) = \begin{pmatrix} \tilde{B}_0(x,\mathcal{B}) & \tilde{A}_0(x,\mathcal{B}) & 0 & 0 & \dots \\ \tilde{B}_1(x,\mathcal{B}) & \tilde{A}_1(x,\mathcal{B}) & \tilde{A}_0(x,\mathcal{B}) & 0 & \dots \\ 0 & \tilde{A}_2(x,\mathcal{B}) & \tilde{A}_1(x,\mathcal{B}) & \tilde{A}_0(x,\mathcal{B}) & \dots \\ 0 & 0 & \tilde{A}_2(x,\mathcal{B}) & \tilde{A}_1(x,\mathcal{B}) & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix}$$

where $\tilde{A}_0(x, \mathcal{B}) = \mathbb{P}(N_n = N_{n-1} + 1, A_n \in \mathcal{B} | A_{n-1} = x)$

- the stationary measure is given by

$$\nu_k(\mathcal{B}) = \int_{\mathcal{E}} \nu_{k-1}(\mathsf{d}x)\tilde{S}(x,\mathcal{B})$$

where $\tilde{S}$ is the minimal nonnegative solution of

$$\begin{aligned}\tilde{S}(x,\mathcal{B}) &= \tilde{A}_0(x,\mathcal{B}) \\ &+ \int_{\mathcal{E}} \tilde{S}(x,\mathsf{d}y)\tilde{A}_1(y,\mathcal{B}) + \int_{\mathcal{E}}\int_{\mathcal{E}} \tilde{S}(x,\mathsf{d}u)\tilde{S}(u,\mathsf{d}y)\tilde{A}_2(y,\mathcal{B})\end{aligned}$$

That is the operator version of

$$R = A_0 + RA_1 + R^2A_2$$

The QBD-RAP at level transitions

$X_n = (N_n, A_n)$

- For the QBD-RAP the kernels are given by

$$\tilde{A}_i(\boldsymbol{a}, \mathcal{B}) = \int_0^\infty \boldsymbol{a} e^{A_1 t} A_i \boldsymbol{e} I_{\mathcal{B}} \left(\frac{\boldsymbol{a} e^{A_1 t} A_i}{\boldsymbol{a} e^{A_1 t} A_i \boldsymbol{e}} \right) \mathrm{d}t \qquad i = 0, 2$$

- We now introduce the expected value of the phase process at level changes into level n

$$\boldsymbol{\nu}_{i+1} = \int_{\mathcal{A}_2} \boldsymbol{x} \nu_{i+1}(\mathrm{d}\boldsymbol{x}) = \int_{\mathcal{A}_2} \boldsymbol{x} \int_{\mathcal{A}_2} \nu_i(\mathrm{d}\boldsymbol{y}) \tilde{S}(\boldsymbol{y}, \mathrm{d}\boldsymbol{x})$$

$$= \int_{\mathcal{A}_2} \int_{\mathcal{A}_2} \nu_i(\mathrm{d}\boldsymbol{y}) \boldsymbol{x} \tilde{S}(\boldsymbol{y}, \mathrm{d}\boldsymbol{x}) = \int_{\mathcal{A}_2} \nu_i(\mathrm{d}\boldsymbol{y}) \bar{S}(\boldsymbol{y})$$

where $\bar{S}(\boldsymbol{y}) = \int_{\mathcal{A}_2} \boldsymbol{x} \tilde{S}(\boldsymbol{y}, \mathrm{d}\boldsymbol{x})$.

We define the successive iterates $\tilde{S}_k(\boldsymbol{a}, B)$ $\tilde{S}_k(\boldsymbol{a}, B) = 0$

$$\tilde{S}_{k+1}(\boldsymbol{a}, B) = \tilde{A}_0(\boldsymbol{a}, B) + \int_{\mathcal{A}_2} \int_{\mathcal{A}_2} \tilde{S}_k(\boldsymbol{a}, \mathsf{d}\boldsymbol{x}) \tilde{S}_k(\boldsymbol{x}, \mathsf{d}\boldsymbol{y}) \tilde{A}_2(\boldsymbol{y}, B)$$

Corresponding to $\bar{S}(\boldsymbol{a})$ we introduce

$$\bar{S}_k(\boldsymbol{a}) = \int_{\mathcal{A}_2} \boldsymbol{x} \tilde{S}_k(\boldsymbol{a}, \mathsf{d}\boldsymbol{x})$$

with $\bar{S}_0(\boldsymbol{a}) = \mathbf{0}$

Lemma 5 *The mean operator $\bar{S}_k(\boldsymbol{a})$ of the k'th iterate $\tilde{S}_k(\boldsymbol{a})$ is linear for all k i.e.*

$$\bar{S}_k(\boldsymbol{a}) = \boldsymbol{a} S_k$$

where

$$S_{k+1} = (-A_1)^{-1} A_0 + S_k^2 (-A_1)^{-1} A_2$$

Operator equation for $\bar{\bar{S}}$

Theorem 6 *The mean operator $\bar{S}(\boldsymbol{a})$ of $\tilde{S}(\boldsymbol{a})$ is linear i.e.*

$$\bar{S}(\boldsymbol{a}) = \boldsymbol{a}S.$$

$$\begin{aligned}
\bar{S}(\boldsymbol{a}) =& \int_{\mathcal{E}} \boldsymbol{x}\tilde{S}(\boldsymbol{a}, \mathrm{d}\boldsymbol{x}) \\
=& \int_{\mathcal{E}} \boldsymbol{x}\tilde{A}_0(\boldsymbol{a}, \mathrm{d}\boldsymbol{x}) + \int_{\mathcal{E}} \boldsymbol{x} \int_{\mathcal{E}} \tilde{S}^{(2)}(\boldsymbol{a}, \mathrm{d}\boldsymbol{y})\tilde{A}_2(\boldsymbol{y}, \mathrm{d}\boldsymbol{x}) \\
=& \boldsymbol{a}(-A_1)^{-1}A_0 + \int_{\mathcal{E}} \boldsymbol{x} \int_{\mathcal{E}} \tilde{S}^{(2)}(\boldsymbol{a}, \mathrm{d}\boldsymbol{y})\tilde{A}_2(\boldsymbol{y}, \mathrm{d}\boldsymbol{x}) \\
=& \boldsymbol{a}(-A_1)^{-1}A_0 \\
&+ \int_{\mathcal{E}} \boldsymbol{x} \int_{\mathcal{E}} \int_{\mathcal{E}} \tilde{S}(\boldsymbol{a}, \mathrm{d}\boldsymbol{u})\tilde{S}(\boldsymbol{u}, \mathrm{d}\boldsymbol{y}) \int_0^\infty \boldsymbol{y}e^{A_1 t}A_2\boldsymbol{e} I_{\mathrm{d}\boldsymbol{x}} \left(\frac{\boldsymbol{y}e^{A_1 t}A_2}{\boldsymbol{y}e^{A_1 t}A_2\boldsymbol{e}} \right) \mathrm{d}t
\end{aligned}$$

$$
\begin{aligned}
\bar{S}(\boldsymbol{a}) &= \boldsymbol{a}(-A_1)^{-1}A_0 \\
&+ \int_{\mathcal{E}}\int_{\mathcal{E}} \tilde{S}(\boldsymbol{a},\mathsf{d}\boldsymbol{u}) \int_0^{\infty} \tilde{S}(\boldsymbol{u},\mathsf{d}\boldsymbol{y})\boldsymbol{y}e^{A_1 t}A_2\boldsymbol{e} \int_{\mathcal{E}} \boldsymbol{x} I_{\mathsf{d}\boldsymbol{x}} \left(\frac{\boldsymbol{y}e^{A_1 t}A_2}{\boldsymbol{y}e^{A_1 t}A_2\boldsymbol{e}}\right) \mathsf{d}t \\
&= \boldsymbol{a}(-A_1)^{-1}A_0 + \int_{\mathcal{E}}\int_{\mathcal{E}} \tilde{S}(\boldsymbol{a},\mathsf{d}\boldsymbol{u})\boldsymbol{y}\tilde{S}(\boldsymbol{u},\mathsf{d}\boldsymbol{y})(-A_1)^{-1}A_2 \\
&= \boldsymbol{a}(-A_1)^{-1}A_0 + \int_{\mathcal{E}} \tilde{S}(\boldsymbol{a},\mathsf{d}\boldsymbol{u})\bar{S}(\boldsymbol{u})(-A_1)^{-1}A_2 \\
&= \boldsymbol{a}(-A_1)^{-1}A_0 + \boldsymbol{a}S^2(-A_1)^{-1}A_2
\end{aligned}
$$

where we use, that $\bar{S}(\boldsymbol{u}) = \boldsymbol{u}S$

$$
\boldsymbol{\nu}_{i+1} = \boldsymbol{\nu}_i S
$$

The time stationary solution

$$\boldsymbol{\pi}_k = c\boldsymbol{\nu}_k(-A_1)^{-1}$$

inserting we get

$$\boldsymbol{\pi}_k(-A_1) = \boldsymbol{\pi}_{k-1}(-A_1)S \qquad \boldsymbol{\pi}_k = \boldsymbol{\pi}_{k-1}R$$

where

$$R = (-A_1)S(-A_1)^{-1} \qquad S = (-A_1)^{-1}R(-A_1)$$

Rather than solving for S we can solve for R

$$(-A_1)^{-1}R(-A_1) = (-A_1)^{-1}A_0+[(-A_1)^{-1}R(-A_1)]^2(-A_1)^{-1}A_2$$

which is equivalent to

$$A_0 + RA_1 + R^2A_2 = 0$$

Example: RAP,MAP/PH,ME/1-queue

Service time distribution:

$$f(x) = \frac{\frac{\lambda}{2}((\lambda x - \epsilon)^2 + a\epsilon^2))}{1 - \epsilon + \frac{1+a}{2}\epsilon^2} e^{-\lambda x},$$

which is an ME distribution of order 3 with $\boldsymbol{\alpha}$ and S given by

$$\boldsymbol{\alpha} = \frac{1}{1 + \frac{1+a}{2}\epsilon^2 - \epsilon}(1, -\epsilon, \frac{1+a}{2}\epsilon^2), \text{ and } S = \begin{bmatrix} -\lambda & \lambda & 0 \\ 0 & -\lambda & \lambda \\ 0 & 0 & -\lambda \end{bmatrix}.$$

which is also in PH whenever $a > 0$.

Example arrival process: Generic ME distribution

With

$$C = \begin{pmatrix} -\lambda_1 & 0 & 0 \\ \frac{(-\lambda_1+\lambda_2-\omega)(\lambda_2^2+\omega^2)}{\lambda_2^2+\omega^2+b\lambda_1\omega} & -\lambda_2 & \frac{(\lambda_2^2+\omega^2+b\lambda_1\lambda_2)\omega}{\lambda_2^2+\omega^2+b\lambda_1\omega} \\ \frac{(-\lambda_1+\lambda_2+\omega)(\lambda_2^2+\omega^2)}{\lambda_2^2+\omega^2+b\lambda_1\lambda_2} & -\frac{(\lambda_2^2+\omega^2+b\lambda_1\omega)\omega}{\lambda_2^2+\omega^2+b\lambda_1\lambda_2} & -\lambda_2 \end{pmatrix},$$

we get $\boldsymbol{g}(t) = e^{Ct}\boldsymbol{e}$

$$\boldsymbol{g}(t) = \begin{bmatrix} \lambda_1 e^{-\lambda_1 t} \\ \frac{\lambda_1(\lambda_2^2+\omega^2)}{\lambda_2^2+\omega^2+b\lambda_1\lambda_2}\left(e^{-\lambda_1 t} + be^{-\lambda_2 t}\sin(\omega t)\right) \\ \frac{\lambda_1(\lambda_2^2+\omega^2)}{\lambda_2^2+\omega^2+b\lambda_1\lambda_2}\left(e^{-\lambda_1 t} + be^{-\lambda_2 t}\cos(\omega t)\right). \end{bmatrix}$$

The distribution is not phase type for $\lambda_1 = \lambda_2$ and for $|b| = 1$.

Arrival process

A RAP(C, D) with

$$D = \begin{pmatrix} \gamma_1 & 0 & 0 & 0 \\ 0 & \gamma_1 & 0 & 0 \\ 0 & 0 & \gamma_1 & 0 \\ 0 & 0 & 0 & \gamma_2 \end{pmatrix}.$$

(i.e. an alternating Poisson process), as a MAP

$$C = \begin{pmatrix} T - \gamma_1 I & \lambda(1-p)\boldsymbol{e}_n \\ \lambda_3 \boldsymbol{\alpha}_2 & -\lambda_3 - \gamma_2 \end{pmatrix}, \qquad D = \begin{pmatrix} \gamma_1 I & 0 \\ 0 & \gamma_2 \end{pmatrix},$$

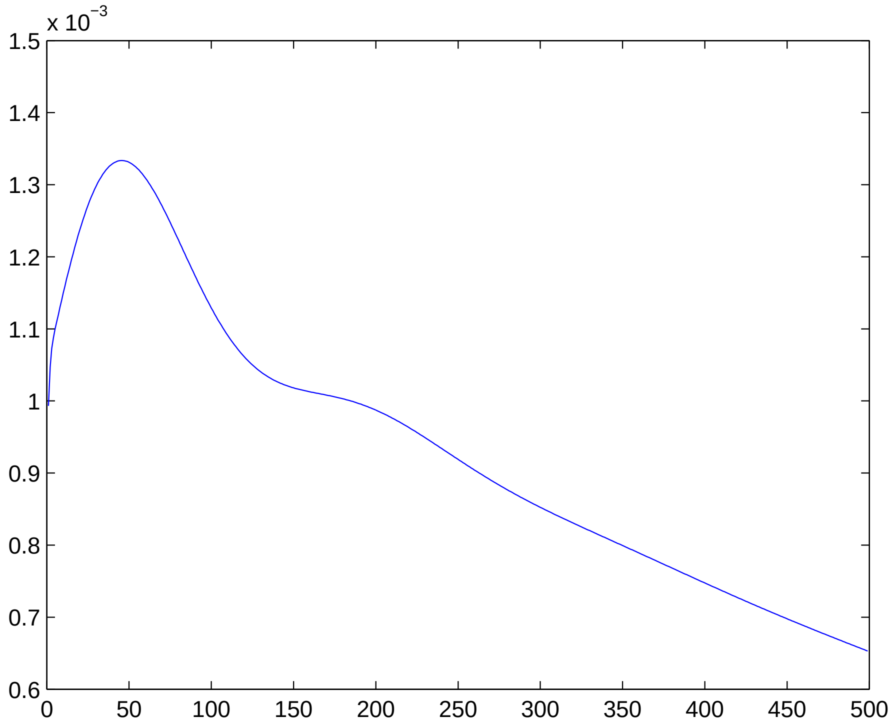
where T and α_2 have dimension

$$n = \frac{2\pi}{\arcsin\left(\frac{2\omega(\lambda_2 - \lambda_1)}{(\lambda_2-\lambda_1)^2+\omega^2}\right)} \in \mathbb{Z}_+,$$

where the expression has to give an integer value.

Experience

- The dimension of the MAP/PH/1 increases linearly with n
- The dimension of the RAP/ME/1 queue is 12
- The results agreed to 10^{-10}



A generalization

- A descriptor of measures corresponding to $\bar{S}(\boldsymbol{a})$
- Operators on measures which has linear effect on the descriptors
- Still miss characterization of matrix equation
  - ◇ Do not invalidate results with respects to algorithms
  - ◇ But could potentially cause numerical instability

Conclusion

- The ME/RAP generalization can be analyzed by solving the matrix equations by Neuts
- So far we have not experienced numerical problems
- We are considering other relevant queues that can be included in the general framework (no candidates yet)