Branching Processes & Queuing Theory

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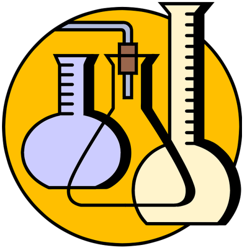
Basic recursion

$$X_{n+1} = \underbrace{\sum_{\ell=1}^{X_n} \zeta_n(\ell)}_{A_n(X_n)} + B_n$$

Applications

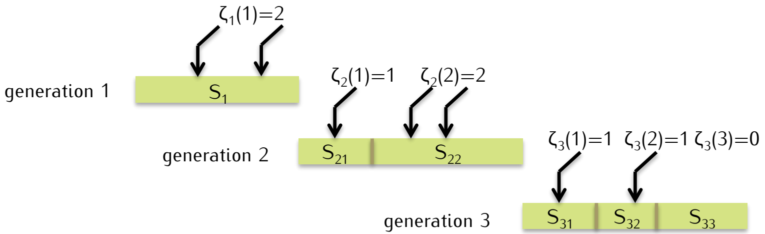
genealogy



chemistry

queuing

Busy periods



Number of customers served in a busy period

= Total number of persons in all generations

$$\Pr[\zeta = n] = \mathrm{E}[I(Pois(\lambda S) = n)]$$

[Kendall, Some Problems in the Theory of Queues, J. Roy. Stat. Soc., 1951.]

Moments

Total number of persons in all generations, no migration

$$X_{n+1} = \sum_{\ell=1}^{X_n} \zeta_n(\ell)\,, \quad X_1 = 1\,, \quad Y = \sum_{n=1}^{\infty} X_n$$

$$\mathsf{E}[X_{n+1}] = \mathsf{E}[X_n]\,\mathsf{E}[\zeta_n] = \prod_{\ell=1}^{n} \mathsf{E}[\zeta_\ell]\,, \quad \mathsf{E}[Y] = 1 + \sum_{n=1}^{\infty} \prod_{\ell=1}^{n} \mathsf{E}[\zeta_\ell]$$

With migration

$$X_{n+1} = \sum_{\ell=1}^{X_n} \zeta_n(\ell) + B_n$$

$$\mathsf{E}[X_{n+1}] = \mathsf{E}[X_n]\,\mathsf{E}[\zeta_n] + \mathsf{E}[B_n]$$

$$\mathsf{E}[X_{n+1}^2] = \mathsf{E}[X_n^2]\,\mathsf{E}[\zeta_n]^2 + \mathsf{E}[X_n](\mathsf{E}[\zeta_n^2] - \mathsf{E}[\zeta_n]^2) + \mathsf{E}[B_n^2] + 2\,\mathsf{E}[B_n]\,\mathsf{E}[X_n]\,\mathsf{E}[\zeta_n]$$

Beyond queuing dynamics ...

Jacques Resing

Functional equations

Soren Asmussen

David Kendall

Generating Functions

Heavy traffic limits

Guy Latouche

Matrix Analytic

Multiple types

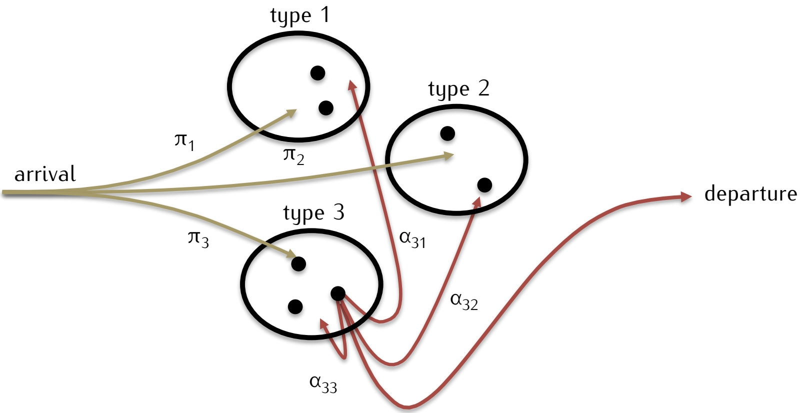
$$\vec{X}_{n+1} = \underbrace{\sum_{k=1}^{K} \sum_{\ell=1}^{X_n(k)} \vec{\zeta}_{n,k}(\ell)}_{\vec{A}_n(\vec{X}_n)} + \vec{B}_n$$

Moments

$$\mathrm{E}[\vec{A}_n(\vec{X}_n)] = \sum_{k=1}^{K} \mathrm{E}[X_n(k)]\,\mathrm{E}[\vec{\zeta}_{n,k}] \doteq \mathcal{A}_n \mathrm{E}[X_n(k)]$$

$$\begin{aligned}
\mathrm{E}[\vec{A}_n(\vec{X}_n)\vec{A}_n(\vec{X}_n)'] &= \sum_{k=1}^{K}\sum_{\ell=1}^{K} \mathrm{E}[X_n(k)X_n(\ell)]\,\mathrm{E}[\vec{\zeta}_{n,k}]\,\mathrm{E}[\vec{\zeta}'_{n,\ell}] \\
&\quad + \sum_{k=1}^{K} \mathrm{E}[X_n(k)](\mathrm{E}[\vec{\zeta}_{n,k}\vec{\zeta}'_{n,k}] - \mathrm{E}[\vec{\zeta}_{n,k}]\,\mathrm{E}[\vec{\zeta}'_{n,k}]) \\
&\doteq \sum_{k=1}^{K} \left(\mathrm{E}[X_n(k)]\,\mathcal{B}_{n,k} + \sum_{\ell=1}^{K} \mathrm{E}[X_n(k)X_n(\ell)]\,\mathcal{C}_{n,k,\ell} \right)
\end{aligned}$$

Discrete-time $M/PH/\infty$ queue



Migration = new arrivals
Types = Phases of the service times
Offspring = At most one, the type being the next phase

[Altman. On stochastic recursive equations and infinite server queues. Infocom 2005]

Queue with $M/PH/\infty$ input

Use the queue content process of the $M/PH/\infty$ as arrival process of another queue

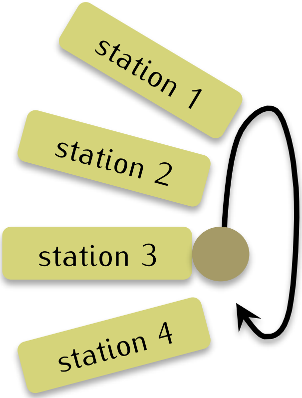
Interpretation: packets produced during "sessions"

Key property: regeneration when there are no arrivals

Generalisation: queues with multi-type Galton-Watson input

[Fiems et al. Queues with Galton-Watson-type arrivals. BWWQT, 2009]

Polling systems



Gated polling
Exhaustive polling
Globally gated polling
Gated-Exhaustive polling

Cyclic routing
Markovian routing

Feedback

Symmetric gated polling

N Stations are polled cyclically. At the nth polling instant, polling stations are ordered as they will be visited.

$X_n^{(1)}$ queue content at the the polling station visited now

$S_{n,k}^{(i)}$ is the number of arrivals at station i during the service time of the kth customer served after the nth polling instant

$T_n^{(i)}$ is the number of arrivals at station i during the switchover time from n to $n+1$

$$X_{n+1}^{(k)} = X_n^{(k+1)} + T_n^{(k+1)} + \sum_{\ell=1}^{X_n^{(1)}} S_{n,\ell}^{(k+1)}$$

$$X_{n+1}^{(N)} = T_n^{(1)} + \sum_{\ell=1}^{X_n^{(1)}} S_{n,\ell}^{(1)}$$

Non-symmetric gated polling

Trick of "renumbering" the stations no longer works

Assume that the server arrives at the mth station at the nth polling time

$$X_{n+1}^{(k)} = X_n^{(k)} + T_n^{(k)} + \sum_{\ell=1}^{X_n^{(m)}} S_{n,m,\ell}^{(k)}$$

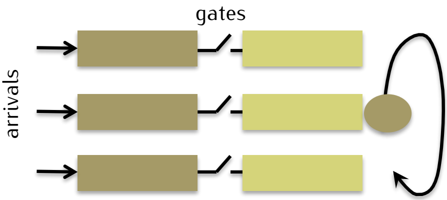
$$X_{n+1}^{(m)} = T_n^{(m)} + \sum_{\ell=1}^{X_n^{(m)}} S_{n,m,\ell}^{(m)}$$

Apply N times as to arrive at the same station $\rightarrow$ still branching with migration!

Gated versus exhaustive

From gated to exhaustive → replace service times by busy periods

From exhaustive to gated → introduce an extra queue



Moving everything from one queue to the other is also a branching process

Is the branching property essential for analysis of the polling system?

Commercial break ...

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Semi-linear processes

Stochastic recursive equation of the form:

$$X_{n+1} = A_n(X_n) + B_n \,,$$

where

1. For some k, $y = y^0 + y^1 + \ldots + y^k$, then $A_n(y)$ can be represented as
$$A_n(y) = \sum_{i=0}^{k} \widehat{A}_n^{(i)}(y^i) \,,$$
with $\widehat{A}_n^{(i)}$ identically distributed with same distribution as A_n.
2. $\mathsf{E}[A_n(y)] = \mathcal{A}y$ and $\mathsf{E}[A_n(y)A_n(y)'] = \sum_{k=1}^{K} \left(y_k \, \mathcal{B}_{n,k} + \sum_{\ell=1}^{K} y_k y_\ell \, \mathcal{C}_{n,k,\ell} \right) \doteq F(yy') + \sum_{k=1}^{K} y_k \, \mathcal{B}_{n,k}$
3. B_n is stationary ergodic

No independence!

Examples

- Linear recurrences (in $\mathbb{R}^N$ or $\mathbb{N}^N$):

$$X_{n+1} = A_n X_n + B_n$$

- Branching in a random environment:

$$X_{n+1} = \sum_{k=1}^{X_n} \zeta_n(k; E_n)$$

- Subordinators $\rightarrow$ recursion for the station times in polling systems

Stability

Theorem

(i) For $n > 0$, Y_n can be written in the form

$$X_n = \widetilde{X}_n + \left(\bigotimes_{i=0}^{n-1} \widehat{A}_i^{(0)} \right) (X_0) \text{ where } \widetilde{X}_n = \sum_{j=0}^{n-1} \left(\bigotimes_{i=n-j}^{n-1} \widehat{A}_i^{(n-j)} \right) (B_{n-j-1})$$

(ii) there is a unique stationary solution X_n^* of $X_n = A_n(X_{n-1}) + B_n$, distributed like

$$X_n^* =_d \sum_{j=0}^{\infty} \left(\bigotimes_{i=n-j}^{n-1} \widehat{A}_i^{(n-j)} \right) (B_{n-j-1}), \qquad n \in \mathbb{Z} \tag{1}$$

The sum on the right side of (1) converges absolutely almost surely and $\lim_{n\to\infty} |X_n - X_n^*| = 0$, almost surely. for any initial value X_0.

Stationary ergodic migration

Correlation of the migration process does not affect means:

$$\mathsf{E}[\vec{X}] = (\mathcal{I} - \mathcal{A})^{-1}\,\mathsf{E}[\vec{B}]$$

But affects the second order moments

Theorem

Assume $\mathsf{E}[\vec{B}] < \infty$, $\mathsf{E}[\vec{B}\vec{B}'] < \infty$ *and* $\lim_{n\to\infty} F^n = 0$, *then* $cov(\vec{X})$ *is the unique solution of*

$$cov(\vec{X}) = cov(B) + \sum_{r=1}^{\infty} \left(\mathcal{A}^r \widehat{\mathcal{B}}(r) + \left[\mathcal{A}^r \widehat{\mathcal{B}}(r)\right]^T\right) + F(cov[\vec{X}]) + \sum_{k=1}^{d} \mathsf{E}[X(k)]\Gamma^{(k)}$$

with $\widehat{\mathcal{B}}(r) = \mathsf{E}[\vec{B}_0 \vec{B}_r'] - \mathsf{E}[\vec{B}]\,\mathsf{E}[\vec{B}]'$

Lifting independence assumption on A_n

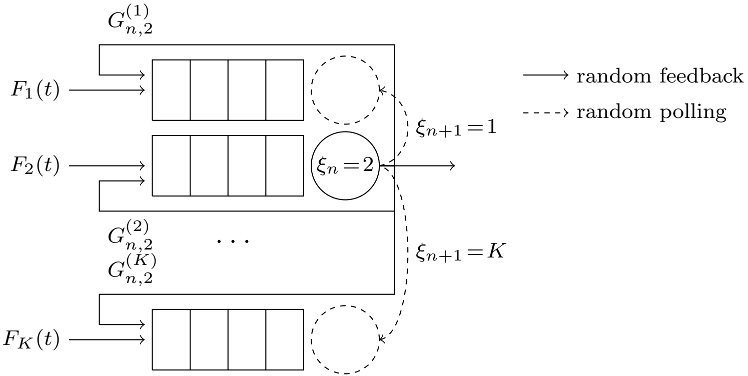
Assume an exogenous Markovian environment Y_n with finite state space

$$\vec{X}_{n+1} = \vec{A}_n(\vec{X}_n, Y_n) + \vec{B}_n(Y_n)$$

The process Z_n is semi-linear

$$\vec{Z}_n = [\vec{X}_n I(Y_n = 1), \vec{X}_n I(Y_n = 2), \ldots, \vec{X}_n I(Y_n = K)]'$$

Back to polling



[Fiems and Altman, Gated polling with stationary ergodic walking times, Markovian routing and random feedback, ANOR, 2012]

Continuous state-space model

- Gated polling policy
- Lévy arrivals (subordinators)
- Semi-linear feedback
- Markovian routing
- Different parameters for every station

Discrete state-space model

- Gated polling policy
- Batch Poisson arrivals, a batch may bring arrivals at different stations
- Poisson feedback during service
- Feedback at the end of service
- Independent service times
- Markovian routing
- Different parameters for every station

Recursion

$$V_{n+1}^{(k)} = V_n^{(k)} + \sum_{i=1}^{V_n^{(\xi_n)}} R_n^{(k)}(i) + G_n^{(k)}\left(\sum_{i=1}^{V_n^{(\xi_n)}} S_n(i)\right)$$

$$+ F_n^{(k)}\left(\sum_{i=1}^{V_n^{(\xi_n)}} S_n(i) + W_n^{(Y_n)}\right), \quad \text{for } k \neq \xi_n,$$

$$V_{n+1}^{(\xi_n)} = \sum_{i=1}^{V_n^{(\xi_n)}} R_n^{(\xi_n)}(i) + G_n^{(\xi_n)}\left(\sum_{i=1}^{V_n^{(\xi_n)}} S_n(i)\right) + F_n^{(\xi_n)}\left(\sum_{i=1}^{V_n^{(\xi_n)}} S_n(i) + W_n^{(Y_n)}\right)$$

Semi-linear framework

$$V_{n+1}^{(\theta(\mathbf{Y}_n))} = \underbrace{\sum_{i=1}^{V_n^{(\theta(\mathbf{Y}_n))}} \left(R_n^{(\theta(\mathbf{Y}_n))}(i) + G_{n,i}^{(\theta(\mathbf{Y}_n))}(S_n(i)) + F_{n,i}^{(\theta(\mathbf{Y}_n))}(S_n(i)) \right)}_{A_n^{(\theta(\mathbf{Y}_n))}(\mathbf{V}_n, Y_n)} + \underbrace{F_{n,0}^{(\theta(\mathbf{Y}_n))}(W_n^{(\mathbf{Y}_n)})}_{B_n^{(\theta(\mathbf{Y}_n))}(\mathbf{Y}_n)} .$$

Open question

For branching processes: "heavy traffic limit to Gamma distributed random variable"

see [Vandermei, Towards a unifying theory on branching-type polling models in heavy traffic. Queueing Systems]

What if semi-linear, and not branching?


questions
Fragen
perguntas
preguntas
domande
vragen