Approximation and error for the waiting time distribution of the $M(AP)/G/1$ queue

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Introduction

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Pollaczek-Khinchine formula

$$\mathbb{P}(W \leq t) = \sum_{n=0}^{\infty} (1-\rho)\rho^n F_0^{*n}(t),$$

where $F_0(t) = \frac{1}{\mathbb{E}S} \int_0^t F^c(u)\, du$ is the stationary service time distribution.

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Solution: Approximations

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$$\mathbb{P}(W > t) \sim \rho e^{-(1-\rho)t/\mathbb{E}S_0}, \quad \mathbb{E}S_0 = \frac{\mathbb{E}S^2}{2\mathbb{E}S}.$$

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$$\mathbb{P}(W > t) \sim \frac{\rho}{1-\rho} F_0^c(t).$$

4. Approximate the stationary service time distribution with a hyperexponential + bound.

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5. **Bonus:** adjust its accuracy!

New idea: The Hill estimator

Theorem

Suppose that $X_j, j \geq 1$ are i.i.d., nonnegative random variables whose common distribution has a regularly varying tail $(F^c(t) = t^{-a}L(t))$, which implies that

$$\frac{k}{n}\mathbb{P}\left[\frac{X_1}{b(n/k)} \in \cdot\right] \xrightarrow{v} v_a(\cdot)$$

in $M_+(0,\infty]$ as $n \to \infty$ and $k = k(n) \to \infty$ with $k/n \to \infty$. Then in $M_+(0,\infty]$,

$$v_n \Rightarrow v_a, \tag{1}$$

where

$$v_a(t,\infty] = t^{-a}, t > 0, a > 0 \quad \& \quad b(t) = F^{\leftarrow}\left(1 - \frac{1}{t}\right).$$

Construction of the "perfect" approximation

Theorem

If (1) holds, then as $n \to \infty$, $k \to \infty$, and $k/n \to 0$,

$$H_{k,n} := \frac{1}{k}\sum_{i=1}^{k} \log \frac{X_{(i)}}{X_{(k+1)}} \xrightarrow{P} \frac{1}{a}.$$

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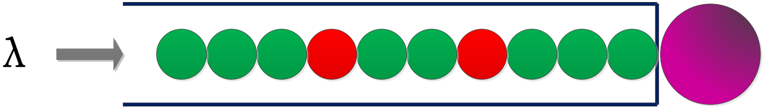

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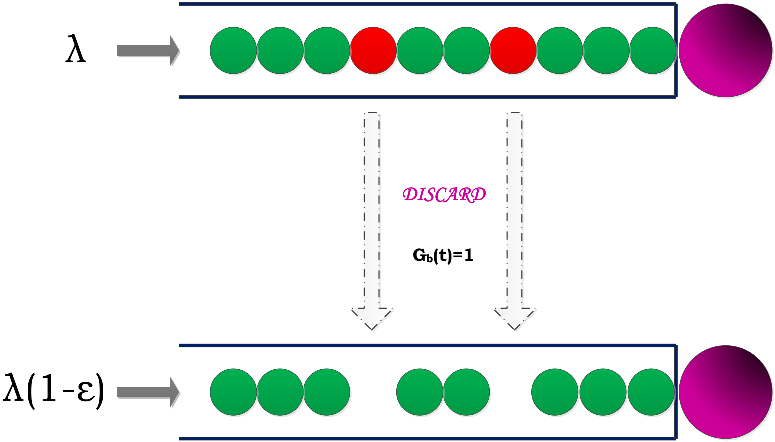

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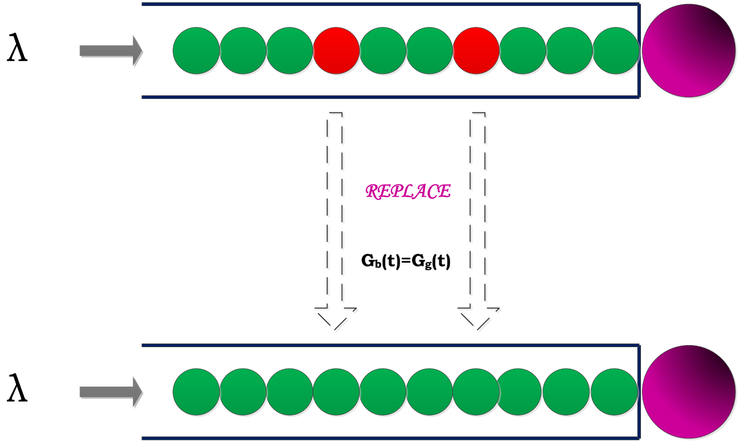

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Laplace Transform of Pollaczek-Khinchine formula

$$\hat{W}_\epsilon(s) = \frac{1-\rho_\epsilon}{1-\rho_\epsilon \hat{F}_0(s)}.$$

$$\hat{W}_\epsilon(s) = \frac{1-(1-\epsilon)\delta-\epsilon\theta}{1-(1-\epsilon)\delta\hat{\beta}^*(s)-\epsilon\theta\hat{\gamma}^*(s)}$$

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- $\delta = \lambda\gamma_g$, and $\theta = \lambda\gamma_b$.

Discard Approximation

$$
\begin{aligned}
\hat{W}_{\epsilon}(s) = & \hat{W}_0(s) \\
& + \sum_{n=1}^{\infty} \left(\frac{\epsilon\theta}{1-\delta+\epsilon\delta} \right)^n \left(\hat{W}_0(s) \right)^{n+1} \left(\hat{\gamma}^*(s) \right)^n \\
& - \sum_{n=1}^{\infty} \left(\frac{\epsilon\theta}{1-\delta+\epsilon\delta} \right)^n \left(\hat{W}_0(s) \right)^{n} \left(\hat{\gamma}^*(s) \right)^{n-1}
\end{aligned}
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Discard Approximation

$$\begin{aligned}\hat{W}_{\epsilon}(s) =& \hat{W}_0(s) \\ &+ \sum_{n=1}^{\infty} \left(\frac{\epsilon\theta}{1-\delta+\epsilon\delta} \right)^n \left(\hat{W}_0(s) \right)^{n+1} \left(\hat{\gamma}^*(s) \right)^n \\ &- \sum_{n=1}^{\infty} \left(\frac{\epsilon\theta}{1-\delta+\epsilon\delta} \right)^n \left(\hat{W}_0(s) \right)^{n} \left(\hat{\gamma}^*(s) \right)^{n-1}\end{aligned}$$

$$\begin{aligned}\stackrel{\mathcal{L}^{-1}}{\Rightarrow} \mathbb{P}(W_{\epsilon} > t) =& \mathbb{P}(W_0 > t) \\ &+ \sum_{n=1}^{\infty} \left(\frac{\epsilon\theta}{1-\delta+\epsilon\delta} \right)^n \left(L_{n+1,n}(t) - L_{n,n-1}(t) \right),\end{aligned}$$

where $L_{n+1,n}(t) = \mathbb{P}(W_{0,1} + \cdots + W_{0,n+1} + C_1^* + \cdots + C_n^* > t)$.

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$$\hat{W}_{\epsilon}(s) = \frac{1-\rho_{\epsilon}}{1-\rho_{\epsilon}\hat{F}_0(s)}.$$

$$\hat{W}_{\epsilon}(s) = \frac{1-\delta-\epsilon(\theta-\delta)}{1-\delta\hat{\beta}^*(s)-\epsilon\left(\theta\hat{\gamma}^*(s)-\delta\hat{\beta}^*(s)\right)}$$

Replace Approximation

$$
\begin{aligned}
\hat{W}_{\epsilon}(s) &= \hat{W}_0(s) \\
&+\theta \sum_{n=1}^{\infty} \left(\frac{\epsilon}{1-\delta}\right)^n \left(\hat{W}_0(s)\right)^{n+1} \hat{\gamma}^*(s) \sum_{k=0}^{n-1} \binom{n-1}{k} (\theta\hat{\gamma}^*(s))^k (-\delta\hat{\beta}^*(s))^{n-1-k} \\
&-\theta \sum_{n=1}^{\infty} \left(\frac{\epsilon}{1-\delta}\right)^n \left(\hat{W}_0(s)\right)^{n} \sum_{k=0}^{n-1} \binom{n-1}{k} (\theta\hat{\gamma}^*(s))^k (-\delta\hat{\beta}^*(s))^{n-1-k} \\
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&+\delta \sum_{n=1}^{\infty} \left(\frac{\epsilon}{1-\delta}\right)^n \left(\hat{W}_0(s)\right)^{n} \sum_{k=0}^{n-1} \binom{n-1}{k} (\theta\hat{\gamma}^*(s))^k (-\delta\hat{\beta}^*(s))^{n-1-k}.
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Replace Approximation

$$\overset{\mathcal{L}^{-1}}{\Rightarrow}$$

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\mathbb{P}(W_\epsilon > t) &= \mathbb{P}(W_0 > t) \\
&+\theta \sum_{n=1}^{\infty} \left(\frac{\epsilon}{1-\delta}\right)^n \sum_{k=0}^{n-1} \binom{n-1}{k} \theta^k (-\delta)^{n-1-k} \left(L_{n+1,k+1,n-1-k}(t) - L_{n,k,n-1-k}(t)\right) \\
&-\delta \sum_{n=1}^{\infty} \left(\frac{\epsilon}{1-\delta}\right)^n \sum_{k=0}^{n-1} \binom{n-1}{k} \theta^k (-\delta)^{n-1-k} \left(L_{n+1,k,n-k}(t) - L_{n,k,n-1-k}(t)\right),
\end{aligned}$$

where

$L_{s,m,r}(t) = \mathbb{P}(W_{0,1} + \cdots + W_{0,s} + C_1^* + \cdots + C_m^* + B_1^* + \cdots + B_r^* > t)$,
$\forall s, m, r \in \mathbb{N}$.

Both approximations

Discard model

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Replace model

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Numerical example

Abate-Whitt, 1999

Bad service time distribution with Laplace transform:
$\hat{G}_b(s) = 1 - \frac{s}{(\mu+\sqrt{s})(1+\sqrt{s})}$, with mean μ^{-1}, all higher moments infinite and $\mu \in \mathbb{R}^+$.

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Probability density function of the stationary service time distribution:

$$f_{C^*}(t) = \left(\frac{\mu}{1-\mu}\right)\left(\psi(t) - \mu^2\psi(\mu^2 t)\right),$$

where $\psi(t) \equiv e^t \frac{2}{\sqrt{\pi}} \int_{\sqrt{t}}^{\infty} e^{-x^2} dx$.

Approximations when $\epsilon = 0.01$ & $\rho = 0.5$.

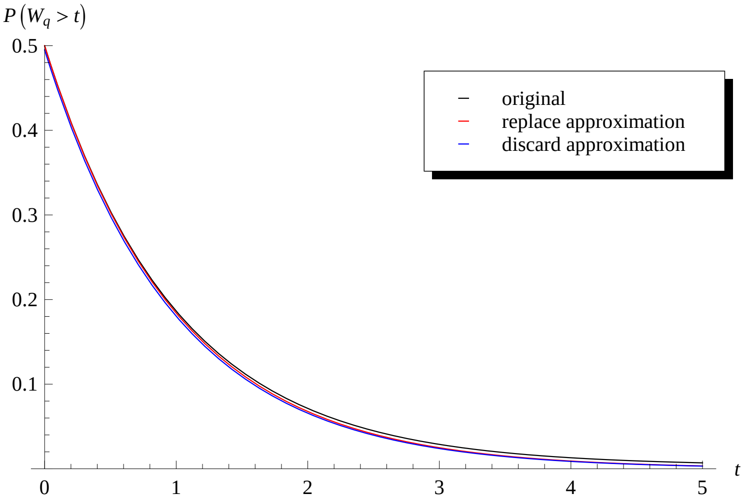

Figure: Exponential 'good' service time distribution with rate 2 and Abate-Whitt 'bad' service time distribution with $\mu = 2$.

Control check for “perfect” approximation

“Perfect”	Discard	Replace
Domain	✓	✓
Error	lower bound	approximation
Requirements	✓	✓
L.T. & H.T.	×	×
Bonus		

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Tail behavior of the exact waiting time distribution

If C is Subexponential, then S, S^* are Subexponentials, too. So,

$$\mathbb{P}(W_\epsilon > t) \sim \frac{\rho_\epsilon}{1-\rho_\epsilon}\mathbb{P}(S^* > t) \sim \frac{\epsilon\theta}{1-\delta+\epsilon\delta-\epsilon\theta}\mathbb{P}(C^* > u).$$

Improved approximations

New idea: add the first term ($n = 1$) of the power series to $\mathbb{P}(W_0 > t)$.

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$$\mathbb{P}(W_{apD} > t) = \mathbb{P}(W_0 > t) + \frac{\epsilon\theta}{1 - \eta\delta}\left(\mathbb{P}(W_{0,1} + W_{0,2} + C^* > t) - \mathbb{P}(W_0 > t)\right),$$

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Replace approximation

$$\begin{aligned}\mathbb{P}(W_{apR>t}) =& \mathbb{P}(W_{0,1} > t) \\ & + \frac{\epsilon\theta}{1-\delta}\left(\mathbb{P}(W_{0,1} + W_{0,2} + C^* > t) - \mathbb{P}(W_0 > t)\right) \\ & - \frac{\epsilon\delta}{1-\delta}\left(\mathbb{P}(W_{0,1} + W_{0,2} + B^* > t) - \mathbb{P}(W_0 > t)\right).\end{aligned}$$

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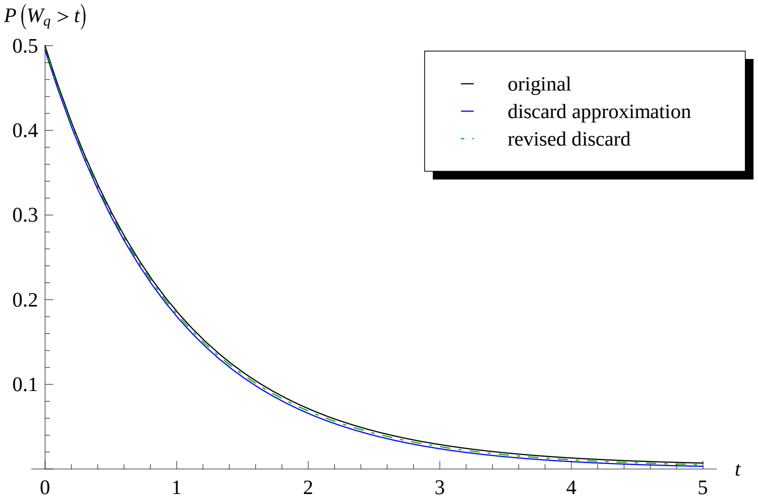

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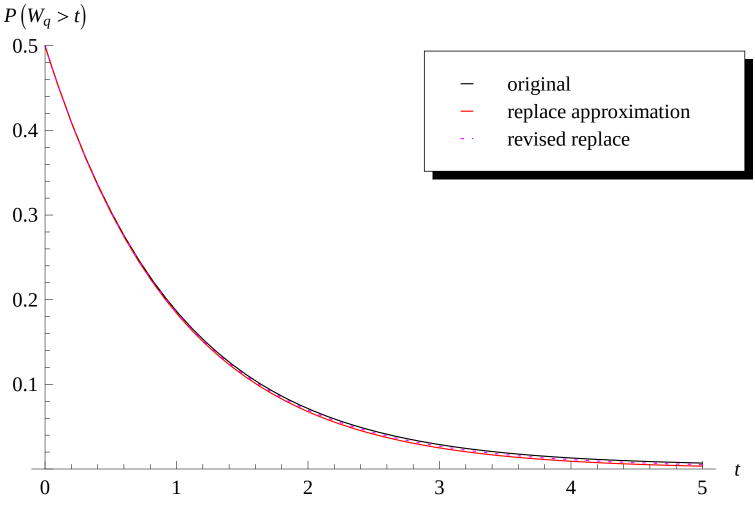
Replace approximations when $\epsilon = 0.01$ & $\rho = 0.5$.



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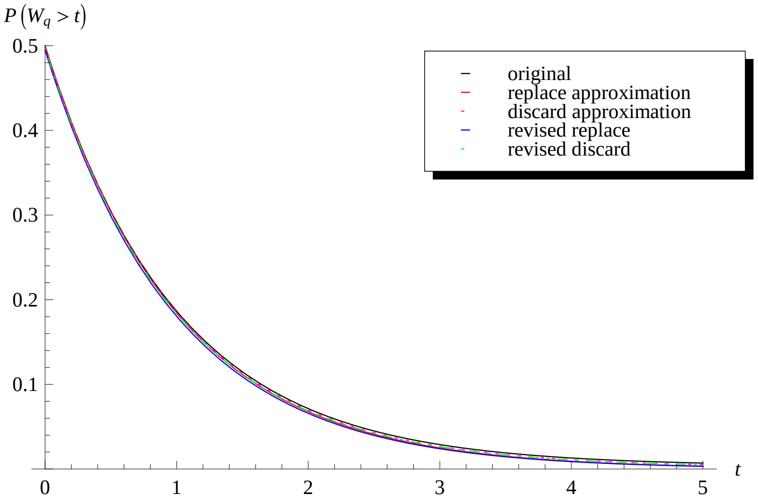

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Thank you for your attention