Extreme values in terms of a matrix exponential: splitting methods in the $M/M/c$ retrial queue

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Talk based on the manuscript
Maximum queue lengths during a fixed time interval in the $M/M/c$ retrial queue, coauthored with A. Gómez-Corral

Organization of the talk

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1. Introduction
2. Maximum queue length during a certain time interval
   - Matrix exponential solution
   - Computation of the matrix exponential solution
   - Accuracy of the solution
3. Numerical results
4. Discussion

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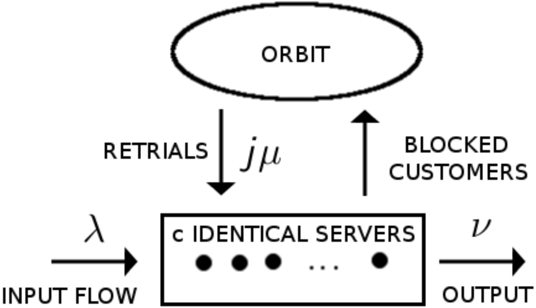
The $M/M/c$ retrial queue

The $M/M/c$ retrial queue is given by:

- λ, Poisson arrival rate
- ν, exponential service rate
- $j\mu$, retrial rate if there are $j \geq 1$ customers in orbit
- c, number of servers



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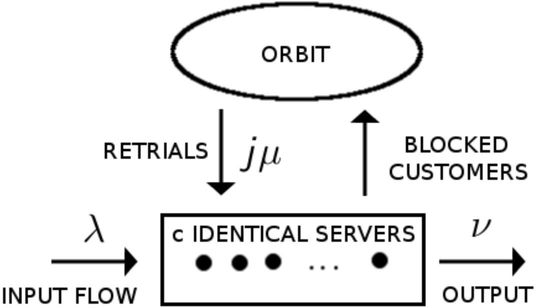

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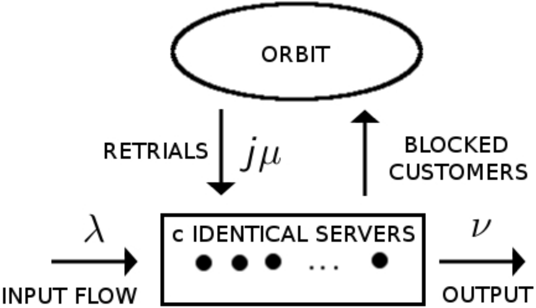

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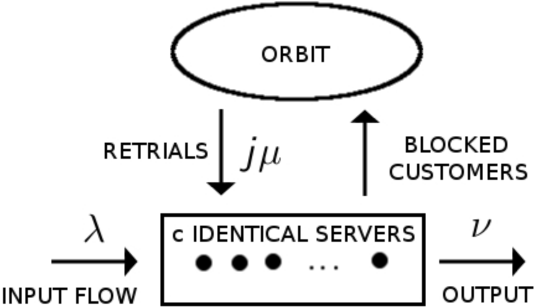
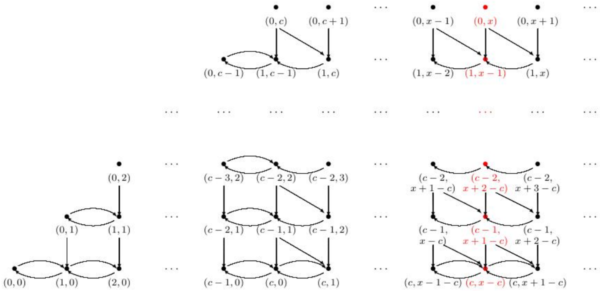


A regular time-homogeneous irreducible CTMC

$$\mathcal{X} \quad = \quad \{X(t) = (C(t), N(t)) : t \geq 0\},$$

on $\mathcal{S} = \{0, ..., c\} \times \{0, 1, 2, ...\}$, where $C(t)$ is the number of busy servers, and $N(t)$ is the number of customers in orbit at time t. If $\mathcal{S} = \cup_{j=0}^{\infty} l(j)$ with $l(j) = \{(i, j - i) : 0 \leq i \leq min(j, c)\}$, then $\mathcal{X}$ is a LD-QBD process



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The quality of service (QoS) is usually measured in terms of classical queueing performance descriptors:

- Expected queue length.
- Waiting time.
- Stationary queue length distribution.

Assuming a stationary regime ($\rho = \lambda/(c\nu) < 1$) is needed to guarantee the existence, and subsequent computation, of most classical queueing measures.

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The distribution of X_{max} is a performance descriptor of practical relevance in the $M/M/c$ retrial queue:

- It is a measure of system congestion.
- It gives support to the adoption of drastic decisions such as an increase of the number of agents or rescheduling of common resources.
- It can be computed even in non-stationary regime, but it might be defective (i.e., $P(X_{max} < \infty) < 1$).

See for example:

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Instead of X_{max}, we may be interested in

$Z(t_0)$: **maximum number of customers in the system (servers + orbit) during a predetermined interval** $[0, t_0]$,

which is a non-defective random variable.

Objective

Computation of

$$P\left(Z(t_0) \leq x \,|\, C(0) = i, N(0) = j\right), \quad \text{for } x \geq i + j \text{ and } (i,j) \in \mathcal{S},$$

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For $x \geq i + j$ and $(i, j) \in \mathcal{S}$, the conditional probability

$$P\left(Z(t_0) > x \,|\, C(0) = i, N(0) = j\right)$$

is equivalent to the probability that, starting from $\overline{X}(0) = (i, j)$, the process $\overline{\mathcal{X}}(x)$ visits the absorbing state $x + 1$ at time t_0.

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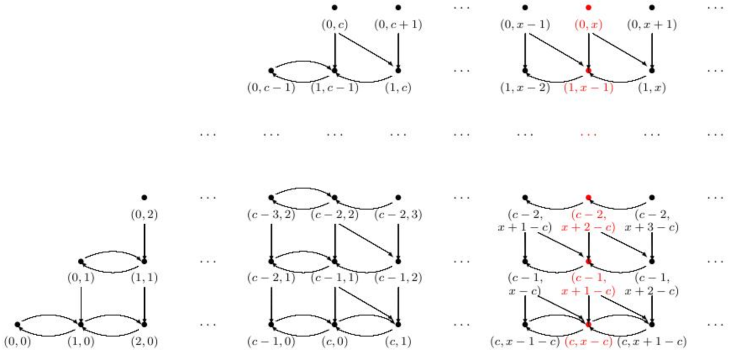

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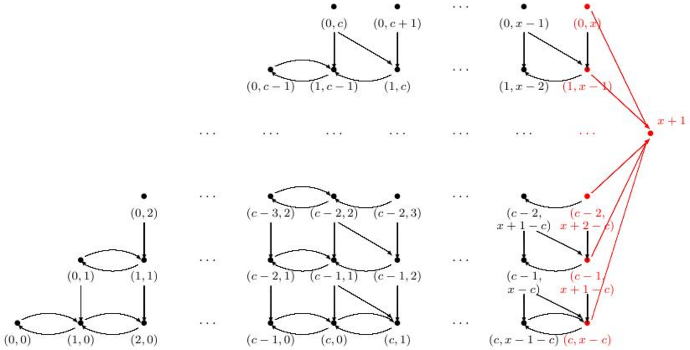

We consider an auxiliary absorbing process $\overline{\mathcal{X}}(x) = \{\overline{X}(t) : t \geq 0\}$ defined on the state space

$$\overline{\mathcal{S}}(x) = \bigcup_{k=0}^{x} l(k) \cup \{x+1\},$$

where the kth level $l(k)$ is given by

$$l(k) = \{(i, k-i) : 0 \leq i \leq \min(k, c)\}, \quad k \geq 0,$$

and the state $x+1$ is obtained by lumping the states of $\cup_{k=x+1}^{\infty} l(k)$ together to make a single absorbing state. Transitions between level $l(x)$ and the state $x+1$ are obtained from those transitions from level $l(x)$ to level $l(x+1)$.

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The infinitesimal generator of $\overline{\mathcal{X}}(x)$ has the form

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and its standard transition function $\overline{\mathbf{P}}(t_0; x)$ can be expressed as

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Therefore, we have

$$P(Z(t_0) \leq x | C(0) = i, N(0) = j) \quad = \quad 1 - \overline{e}_{J(x)}(i,j)\mathbf{p}_{x+1}(t_0; x),$$

where $\overline{e}_{J(x)}(i,j)$ is a row vector of order $J(x)$ such that all its entries are equal to 0, except for the entry associated with the state (i,j) which is equal to 1. Since $\overline{\mathbf{P}}(t_0; x) = \exp\{\overline{\mathbf{Q}}(x)t_0\}$,

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In this expression,

- The matrix $(-\mathbf{T}^{-1}(x))\mathbf{t}_{x+1}(x) = \mathbf{e}_{J(x)}$ consists of the conditional probabilities that the absorption into $x+1$ occurs in a finite time.
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In principle, $\exp\{\mathbf{T}(x)t_0\}$ could be computed in many ways:

- Series methods: Taylor series, Padé approximation, scaling and squaring, Chebyshev rational approximation
- Ordinary differential equation methods: general purpose O.D.E. solver, single step / multistep O.D.E. solvers
- Polynomial methods: Cayley-Hamilton, Lagrange interpolation, Newton interpolation, Vandermonde, inverse Laplace transforms, companion matrix
- Matrix decomposition methods: eigenvectors, triangular systems of eigenvectors, Jordan canonical form, Schur, block diagonal
- Splitting methods

An interesting survey is the paper by Moler and Van Loan (2003)
Software: MathLab, Mathematica, Mapple, ISML library, etc.

In practice, consideration of computational stability and efficiency indicates that some of the methods are preferable to others, but that none are completely satisfactory when they are implemented as general-purpose algorithms

For a value $x \geq i + j$, the dimension of $\exp\{\mathbf{T}(x)t_0\}$ is given by $J(x) = \frac{(x+1)(x+2)}{2}$ if $1 \leq x \leq c$, and $\frac{(c+1)(c+2)}{2} + (x - c)(c + 1)$ if $x \geq c + 1$.

As a result,

- Increasing values of x will imply more demanding memory requirements
- General-purpose algorithms will fail to give satisfactory results as x progressively increases

Splitting methods

For a certain splitting $\mathbf{T}(x) = \mathbf{U}(x) + \mathbf{V}(x)$, it is known that

$$\exp\{\mathbf{T}(x)t_0\} = \exp\{\mathbf{U}(x)t_0\}\exp\{\mathbf{V}(x)t_0\}$$

if and only if $\mathbf{U}(x)$ and $\mathbf{V}(x)$ commute

As $\mathbf{U}(x)$ and $\mathbf{V}(x)$ do not commute, the exponentials of the matrices $\mathbf{U}(x)$ and $\mathbf{V}(x)$ are directly related to that of $\mathbf{T}(x)$ by

$$\exp\{\mathbf{T}(x)t_0\} = \lim_{p\to\infty}\left(\exp\left\{\mathbf{U}(x)\frac{t_0}{p}\right\}\exp\left\{\mathbf{V}(x)\frac{t_0}{p}\right\}\right)^p.$$

Moller and Van Loan (2003) suggest the approximation

$$\exp\{\mathbf{T}(x)t_0\} \simeq \left(\exp\{\mathbf{U}(x)t\}\exp\{\mathbf{V}(x)t\}\right)^{p_0},$$

where $t = p_0^{-1}t_0$, for an appropriately selected integer p_0.

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As $\mathbf{U}(x)$ and $\mathbf{V}(x)$ do not commute, the exponentials of the matrices $\mathbf{U}(x)$ and $\mathbf{V}(x)$ are directly related to that of $\mathbf{T}(x)$ by

$$\exp\{\mathbf{T}(x)t_0\} = \lim_{p\to\infty}\left(\exp\left\{\mathbf{U}(x)\frac{t_0}{p}\right\}\exp\left\{\mathbf{V}(x)\frac{t_0}{p}\right\}\right)^p .$$

Moller and Van Loan (2003) suggest the approximation

$$\exp\{\mathbf{T}(x)t_0\} \simeq (\exp\{\mathbf{U}(x)t\}\exp\{\mathbf{V}(x)t\})^{p_0} ,$$

where $t = p_0^{-1}t_0$, for an appropriately selected integer p_0.

$exp\{\mathbf{T}(x)t_0\}$

- **Case 1:** $1 \le x \le c$.
- **Case 2:** $x \ge c+1$.

Case 1: $1 \le x \le c$.

$\mathbf{T}(x) = \mathbf{U}(x) + \mathbf{V}(x)$ is defined by

$$\mathbf{U}(x) = \begin{pmatrix} \mathbf{A}_{00} & \mathbf{A}_{01} & & & \\ & \mathbf{A}_{11} & \mathbf{A}_{12} & & \\ & & \ddots & \ddots & \\ & & & \mathbf{A}_{x-1,x-1} & \mathbf{A}_{x-1,x} \\ & & & & \mathbf{A}_{xx} \end{pmatrix},$$

$$\mathbf{V}(x) = \begin{pmatrix} 0 & & & & \\ \mathbf{A}_{10} & & & & \\ & \mathbf{A}_{21} & & & \\ & & \ddots & & \\ & & & \mathbf{A}_{x,x-1} & \mathbf{0}_{(x+1)\times(x+1)} \end{pmatrix}.$$

$exp\{\mathbf{T}(x)t_0\}$

- **Case 1:** $1 \leq x \leq c$.
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Case 1: $1 \leq x \leq c$.
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$$\mathbf{V}(x) = \begin{pmatrix} 0 & & & & \\ \mathbf{A}_{10} & & & & \\ & \mathbf{A}_{21} & & & \\ & & \ddots & & \\ & & & \mathbf{A}_{x,x-1} & \mathbf{0}_{(x+1)\times(x+1)} \end{pmatrix}.$$

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$$\mathbf{V}(x) = \begin{pmatrix} 0 & & & & \\ \mathbf{A}_{10} & & & & \\ & \mathbf{A}_{21} & & & \\ & & \ddots & & \\ & & & \mathbf{A}_{x,x-1} & \mathbf{0}_{(x+1)\times(x+1)} \end{pmatrix}.$$

For $1 \le x \le c$, it is seen that

$$\exp\{\mathbf{V}(x)t\} = \sum_{k=0}^{x} \frac{t^k}{k!}\mathbf{V}^k(x)$$

$$= \left(\begin{array}{ccccc|c} \mathbf{M}(0;0) & & & & & \\ \mathbf{M}(1;0) & \mathbf{M}(1;1) & & & & \\ \vdots & \vdots & \ddots & & & \\ \mathbf{M}(x-1;0) & \mathbf{M}(x-1;1) & \cdots & \mathbf{M}(x-1;x-1) & & \\ \hline \mathbf{M}(x;0) & \mathbf{M}(x;1) & \cdots & \mathbf{M}(x;x-1) & & \mathbf{M}(x;x) \end{array}\right),$$

where

$$\mathbf{M}(y;y') = \begin{cases} \mathbf{I}_{y+1} & \text{if } 1 \le y \le x, \\ \mathbf{A}_{y,y-1} \cdot \ldots \cdot \mathbf{A}_{y'+1,y'}, & \text{if } 0 \le y' \le y-1, 1 \le y \le x. \end{cases}$$

For $1 \leq x \leq c$, it is seen that

$$\exp\{\mathbf{V}(x)t\} = \sum_{k=0}^{x} \frac{t^k}{k!}\mathbf{V}^k(x)$$

$$= \left(\begin{array}{cccc|c} \mathbf{M}(0;0) & & & & \\ \mathbf{M}(1;0) & \mathbf{M}(1;1) & & & \\ \vdots & \vdots & \ddots & & \\ \mathbf{M}(x-1;0) & \mathbf{M}(x-1;1) & \cdots & \mathbf{M}(x-1;x-1) & \\ \hline \mathbf{M}(x;0) & \mathbf{M}(x;1) & \cdots & \mathbf{M}(x;x-1) & \mathbf{M}(x;x) \end{array}\right),$$

where

$$\mathbf{M}(y;y') = \begin{cases} \mathbf{I}_{y+1} & \text{if } 1 \leq y \leq x, \\ \mathbf{A}_{y,y-1} \cdot \ldots \cdot \mathbf{A}_{y'+1,y'}, & \text{if } 0 \leq y' \leq y-1, 1 \leq y \leq x. \end{cases}$$

Technical condition: For $1 \leq x \leq c$, the eigenvalues of the matrix $\mathbf{U}(x)$ are all distinct if and only if

- (A.1) [Eigenvalues of the sub-matrix $\mathbf{A}_{yy}$ are all distinct]:

$$\nu \neq \mu$$

- (A.2) [Eigenvalues of sub-matrices $\mathbf{A}_{yy}$ and $\mathbf{A}_{y'y'}$ with $y < y'$ are also distinct]:

$$\nu \neq \left(1 + \frac{y' - y}{l - l'}\right)\mu,$$

  for every pair (y, y') of integers with $0 \leq y < y' \leq x$, and integers $0 \leq l \leq y$ and $0 \leq l' \leq y'$ with $l \neq l'$

Under the technical condition, we derive the decomposition formula $\mathbf{U}(x) = \mathbf{R}_x \mathrm{diag}[r(0;0), ..., r(x;x)]\mathbf{R}_x^{-1}$, which implies

$$\exp\{\mathbf{U}(x)t\} = \mathbf{R}_x \begin{pmatrix} e^{r(0;0)t} & & & & \\ & e^{r(1;0)t} & & & \\ & & e^{r(1;1)t} & & \\ & & & \ddots & \\ & & & & e^{r(x;x)t} \end{pmatrix} \mathbf{R}_x^{-1},$$

where $r(y;l) = -(\lambda + l\nu + (y-l)\mu)$ and $\mathbf{R}_x$ consists of the right eigenvectors of $\mathbf{U}(x)$ associated with the eigenvalues $r(y;l)$, for $0 \leq l \leq y \leq x$.

The matrix $\mathbf{R}_x$ has the structured form

$$\mathbf{R}_x = \left(\begin{array}{cccccc|c} \mathbf{P}_{00} & \mathbf{P}_{01} & \mathbf{P}_{02} & \cdots & \mathbf{P}_{0,x-1} & & \mathbf{P}_{0x} \\ & \mathbf{P}_{11} & \mathbf{P}_{12} & \cdots & \mathbf{P}_{1,x-1} & & \mathbf{P}_{1x} \\ & & \mathbf{P}_{22} & \cdots & \mathbf{P}_{2,x-1} & & \mathbf{P}_{2x} \\ & & & \ddots & \vdots & & \vdots \\ & & & & \mathbf{P}_{x-1,x-1} & & \mathbf{P}_{x-1,x} \\ \hline & & & & & & \mathbf{P}_{xx} \end{array}\right) = \left(\begin{array}{c|c} \mathbf{R}_{x-1} & \mathbf{N}_x \\ \hline & \mathbf{P}_{xx} \end{array}\right),$$

where the columns in $\mathbf{P}_{y'y} = [\mathbf{p}(y',y;0), \mathbf{p}(y',y;1), ..., \mathbf{p}(y',y;y)]$ are given by

$$\mathbf{p}(y',y;l) = \begin{cases} \mathbf{v}(y;l), & \text{if } y' = y, \\ \prod_{k=y'}^{y-1} \left(r(y;l)\mathbf{I}_{k+1} - \mathbf{A}_{kk}\right)^{-1} \mathbf{A}_{k,k+1}\mathbf{v}(y;l), & \text{if } 0 \le y' \le y-1. \end{cases}$$

and $\mathbf{v}(y;l)$ denotes the right eigenvector of $\mathbf{A}_{yy}$ associated with the eigenvalue $r(y;l) = -(\lambda + l\nu + (y-l)\mu)$.

$$\left(r(y;l)\mathbf{I}_{k+1} - \mathbf{A}_{kk}\right)^{-1} = \sum_{l'=0}^{k} \frac{\mathbf{v}(k;l')\mathbf{w}(k;l')}{r(y;l) - r(k;l')}, \quad 1 \le k \le c.$$

The matrix $\mathbf{R}_x$ has the structured form

$$\mathbf{R}_x = \left(\begin{array}{cccccc|c} \mathbf{P}_{00} & \mathbf{P}_{01} & \mathbf{P}_{02} & \cdots & \mathbf{P}_{0,x-1} & & \mathbf{P}_{0x} \\ & \mathbf{P}_{11} & \mathbf{P}_{12} & \cdots & \mathbf{P}_{1,x-1} & & \mathbf{P}_{1x} \\ & & \mathbf{P}_{22} & \cdots & \mathbf{P}_{2,x-1} & & \mathbf{P}_{2x} \\ & & & \ddots & \vdots & & \vdots \\ & & & & \mathbf{P}_{x-1,x-1} & & \mathbf{P}_{x-1,x} \\ \hline & & & & & & \mathbf{P}_{xx} \end{array}\right) = \left(\begin{array}{c|c} \mathbf{R}_{x-1} & \mathbf{N}_x \\ \hline & \mathbf{P}_{xx} \end{array}\right),$$

where the columns in $\mathbf{P}_{y'y} = [\mathbf{p}(y', y; 0), \mathbf{p}(y', y; 1), ..., \mathbf{p}(y', y; y)]$ are given by

$$\mathbf{p}(y', y; l) = \begin{cases} \mathbf{v}(y; l), & \text{if } y' = y, \\ \prod_{k=y'}^{y-1} \left(r(y; l)\mathbf{I}_{k+1} - \mathbf{A}_{kk}\right)^{-1} \mathbf{A}_{k,k+1}\mathbf{v}(y; l), & \text{if } 0 \leq y' \leq y-1. \end{cases}$$

and $\mathbf{v}(y; l)$ denotes the right eigenvector of $\mathbf{A}_{yy}$ associated with the eigenvalue $r(y; l) = -(\lambda + l\nu + (y-l)\mu)$.

$$\left(r(y; l)\mathbf{I}_{k+1} - \mathbf{A}_{kk}\right)^{-1} = \sum_{l'=0}^{k} \frac{\mathbf{v}(k; l')\mathbf{w}(k; l')}{r(y; l) - r(k; l')}, \quad 1 \leq k \leq c.$$

Right eigenvector $\mathbf{v}(y;l)$ of $\mathbf{A}_{yy}$ with $0 \le y \le c$ associated with the eigenvalue $r(y;l) = -(\lambda + l\nu + (y-l)\mu)$, for $0 \le l \le y$:

$$v_{l'}(y;l) = \begin{cases} \binom{y-l'}{y-l}\left(1-\frac{\nu}{\mu}\right)^{-(l-l')}, & \text{if } 0 \le l' \le l-1, \\ 1, & \text{if } l' = l, \\ 0, & \text{if } l+1 \le l' \le y. \end{cases}$$

Left eigenvector $\mathbf{w}(y;l)$ of $\mathbf{A}_{yy}$ with $0 \le y \le c$, associated with the eigenvalue $r(y;l) = -(\lambda + l\nu + (y-l)\mu)$, for $0 \le l \le y$:

$$w_{l'}(y;l) = \begin{cases} 0, & \text{if } 0 \le l' \le l-1, \\ 1, & \text{if } l' = l, \\ \binom{y-l}{y-l'}\left(\frac{\nu}{\mu}-1\right)^{-(l'-l)}, & \text{if } l+1 \le l' \le y. \end{cases}$$

Starting with $\exp\{\mathbf{U}(0)t\} = e^{-\lambda t}$, the matrix exponential $\exp\{\mathbf{U}(x)t\}$ is evaluated from

$$\exp\{\mathbf{U}(x)t\} = \begin{pmatrix} \exp\{\mathbf{U}(x-1)t\} & \mathbf{N}_x(t) \\ \mathbf{0}_{(x+1)\times J(x-1)} & \exp\{\mathbf{A}_{xx}t\} \end{pmatrix}, \quad 1 \le x \le c,$$

where $\exp\{\mathbf{A}_{xx}t\} = \mathbf{P}_{xx}\mathbf{T}_{xx}(t)\mathbf{P}_{xx}^{-1}$ with

$$\begin{aligned} \mathbf{N}_x(t) &= \left(\mathbf{N}_x\mathbf{T}_{xx}(t) - \exp\{\mathbf{U}(x-1)t\}\mathbf{N}_x\right)\mathbf{P}_{xx}^{-1}, \\ \mathbf{T}_{xx}(t) &= \mathrm{diag}[e^{r(x;0)t}, e^{r(x;1)t}, ..., e^{r(x;x)t}], \\ \mathbf{P}_{xx} &= [\mathbf{v}(x;0), \mathbf{v}(x;1), ..., \mathbf{v}(x;x)], \\ \mathbf{P}_{xx}^{-1} &= \begin{pmatrix} c^{-1}(x;0)\mathbf{w}(x;0) \\ c^{-1}(x;1)\mathbf{w}(x;1) \\ \vdots \\ c^{-1}(x;x)\mathbf{w}(x;x) \end{pmatrix}, \end{aligned}$$

and $c(x;l) = \sum_{k=0}^{x}(\mathbf{w}(x;l))_k(\mathbf{v}(x;l))_k$.

Starting with $\exp\{\mathbf{U}(0)t\} = e^{-\lambda t}$, the matrix exponential $\exp\{\mathbf{U}(x)t\}$ is evaluated from

$$\exp\{\mathbf{U}(x)t\} = \begin{pmatrix} \exp\{\mathbf{U}(x-1)t\} & \mathbf{N}_x(t) \\ \mathbf{0}_{(x+1)\times J(x-1)} & \exp\{\mathbf{A}_{xx}t\} \end{pmatrix}, \quad 1 \le x \le c,$$

where $\exp\{\mathbf{A}_{xx}t\} = \mathbf{P}_{xx}\mathbf{T}_{xx}(t)\mathbf{P}_{xx}^{-1}$ with

$$\begin{aligned}
\mathbf{N}_x(t) &= \left(\mathbf{N}_x\mathbf{T}_{xx}(t) - \exp\{\mathbf{U}(x-1)t\}\mathbf{N}_x\right)\mathbf{P}_{xx}^{-1}, \\
\mathbf{T}_{xx}(t) &= \mathrm{diag}[e^{r(x;0)t}, e^{r(x;1)t}, ..., e^{r(x;x)t}], \\
\mathbf{P}_{xx} &= [\mathbf{v}(x;0), \mathbf{v}(x;1), ..., \mathbf{v}(x;x)], \\
\mathbf{P}_{xx}^{-1} &= \begin{pmatrix} c^{-1}(x;0)\mathbf{w}(x;0) \\ c^{-1}(x;1)\mathbf{w}(x;1) \\ \vdots \\ c^{-1}(x;x)\mathbf{w}(x;x) \end{pmatrix},
\end{aligned}$$

and $c(x;l) = \sum_{k=0}^{x}(\mathbf{w}(x;l))_k(\mathbf{v}(x;l))_k$.

Starting with $\exp\{\mathbf{U}(0)t\} = e^{-\lambda t}$, the matrix exponential $\exp\{\mathbf{U}(x)t\}$ is evaluated from

$$\exp\{\mathbf{U}(x)t\} = \begin{pmatrix} \exp\{\mathbf{U}(x-1)t\} & \mathbf{N}_x(t) \\ \mathbf{0}_{(x+1)\times J(x-1)} & \exp\{\mathbf{A}_{xx}t\} \end{pmatrix}, \quad 1 \le x \le c,$$

where $\exp\{\mathbf{A}_{xx}t\} = \mathbf{P}_{xx}\mathbf{T}_{xx}(t)\mathbf{P}_{xx}^{-1}$ with

$$\begin{aligned} \mathbf{N}_x(t) &= \left(\mathbf{N}_x\mathbf{T}_{xx}(t) - \exp\{\mathbf{U}(x-1)t\}\mathbf{N}_x\right)\mathbf{P}_{xx}^{-1}, \\ \mathbf{T}_{xx}(t) &= \text{diag}[e^{r(x;0)t}, e^{r(x;1)t}, \ldots, e^{r(x;x)t}], \\ \mathbf{P}_{xx} &= [\mathbf{v}(x;0), \mathbf{v}(x;1), \ldots, \mathbf{v}(x;x)], \\ \mathbf{P}_{xx}^{-1} &= \begin{pmatrix} c^{-1}(x;0)\mathbf{w}(x;0) \\ c^{-1}(x;1)\mathbf{w}(x;1) \\ \vdots \\ c^{-1}(x;x)\mathbf{w}(x;x) \end{pmatrix}, \end{aligned}$$

and $c(x;l) = \sum_{k=0}^{x}(\mathbf{w}(x;l))_k(\mathbf{v}(x;l))_k$.

Starting with $\exp\{\mathbf{U}(0)t\} = e^{-\lambda t}$, the matrix exponential $\exp\{\mathbf{U}(x)t\}$ is evaluated from

$$\exp\{\mathbf{U}(x)t\} = \begin{pmatrix} \exp\{\mathbf{U}(x-1)t\} & \mathbf{N}_x(t) \\ \mathbf{0}_{(x+1)\times J(x-1)} & \exp\{\mathbf{A}_{xx}t\} \end{pmatrix}, \quad 1 \le x \le c,$$

where $\exp\{\mathbf{A}_{xx}t\} = \mathbf{P}_{xx}\mathbf{T}_{xx}(t)\mathbf{P}_{xx}^{-1}$ with

$$\begin{aligned} \mathbf{N}_x(t) &= \left(\mathbf{N}_x\mathbf{T}_{xx}(t) - \exp\{\mathbf{U}(x-1)t\}\mathbf{N}_x\right)\mathbf{P}_{xx}^{-1}, \\ \mathbf{T}_{xx}(t) &= \mathrm{diag}[e^{r(x;0)t}, e^{r(x;1)t}, \ldots, e^{r(x;x)t}], \\ \mathbf{P}_{xx} &= [\mathbf{v}(x;0), \mathbf{v}(x;1), \ldots, \mathbf{v}(x;x)], \\ \mathbf{P}_{xx}^{-1} &= \begin{pmatrix} c^{-1}(x;0)\mathbf{w}(x;0) \\ c^{-1}(x;1)\mathbf{w}(x;1) \\ \vdots \\ c^{-1}(x;x)\mathbf{w}(x;x) \end{pmatrix}, \end{aligned}$$

and $c(x;l) = \sum_{k=0}^{x}(\mathbf{w}(x;l))_k(\mathbf{v}(x;l))_k$.

Case 2: $x \geq c+1$.
$\mathbf{T}(x) = \mathbf{U}(x) + \mathbf{V}(x)$ is defined by

$$\mathbf{U}(x) = \left(\begin{array}{ccccc|cccc} \mathbf{A}_{00} & \mathbf{A}_{01} & & & & & & & \\ & \mathbf{A}_{11} & \mathbf{A}_{12} & & & & & & \\ & & \ddots & \ddots & & & & & \\ & & & \mathbf{A}_{c-1,c-1} & \mathbf{A}_{c-1,c} & & & & \\ & & & & \mathbf{A}_{cc} & \mathbf{A}_{c,c+1} & & & \\ \hline & & & & & \mathbf{A}^*_{c+1,c+1} & \mathbf{A}_{c+1,c+2} & & \\ & & & & & & \ddots & \ddots & \\ & & & & & & & \mathbf{A}^*_{x-1,x-1} & \mathbf{A}_{x-1,c} \\ & & & & & & & & \mathbf{A}^*_{x,x} \end{array}\right),$$

$$\mathbf{V}(x) = \left(\begin{array}{ccccc|cccc} 0 & & & & & & & & \\ \mathbf{A}_{10} & & & & & & & & \\ & \mathbf{A}_{21} & & & & & & & \\ & & \ddots & & & & & & \\ & & & \mathbf{A}_{c,c-1} & \mathbf{0}_{(c+1)\times(c+1)} & & & & \\ \hline & & & & \mathbf{A}_{c+1,c} & \overline{\mathbf{A}}^*_{c+1,c+1} & & & \\ & & & & & \mathbf{A}_{c+2,c+1} & \overline{\mathbf{A}}^*_{c+2,c+2} & & \\ & & & & & & \ddots & \ddots & \\ & & & & & & & \mathbf{A}_{x,x-1} & \overline{\mathbf{A}}^*_{xx} \end{array}\right),$$

Case 2: $x \geq c+1$.
$\mathbf{T}(x) = \mathbf{U}(x) + \mathbf{V}(x)$ is defined by

$$\mathbf{U}(x) = \left(\begin{array}{ccccc|ccccc}
\mathbf{A}_{00} & \mathbf{A}_{01} & & & & & & & & \\
 & \mathbf{A}_{11} & \mathbf{A}_{12} & & & & & & & \\
 & & \ddots & \ddots & & & & & & \\
 & & & \mathbf{A}_{c-1,c-1} & \mathbf{A}_{c-1,c} & & & & & \\
 & & & & \mathbf{A}_{cc} & \mathbf{A}_{c,c+1} & & & & \\
\hline
 & & & & & \mathbf{A}^*_{c+1,c+1} & \mathbf{A}_{c+1,c+2} & & & \\
 & & & & & & \ddots & \ddots & & \\
 & & & & & & & \mathbf{A}^*_{x-1,x-1} & \mathbf{A}_{x-1,c} & \\
 & & & & & & & & \mathbf{A}^*_{x,x} &
\end{array}\right),$$

$$\mathbf{V}(x) = \left(\begin{array}{ccccc|ccccc}
0 & & & & & & & & & \\
\mathbf{A}_{10} & & & & & & & & & \\
 & \mathbf{A}_{21} & & & & & & & & \\
 & & \ddots & & & & & & & \\
 & & & \mathbf{A}_{c,c-1} & \mathbf{0}_{(c+1)\times(c+1)} & & & & & \\
\hline
 & & & & \mathbf{A}_{c+1,c} & \overline{\mathbf{A}}^*_{c+1,c+1} & & & & \\
 & & & & & \mathbf{A}_{c+2,c+1} & \overline{\mathbf{A}}^*_{c+2,c+2} & & & \\
 & & & & & & \ddots & \ddots & & \\
 & & & & & & & \mathbf{A}_{x,x-1} & \overline{\mathbf{A}}^*_{xx} &
\end{array}\right),$$

Case 2: $x \geq c+1$.
$\mathbf{T}(x) = \mathbf{U}(x) + \mathbf{V}(x)$ is defined by

$$\mathbf{U}(x) = \left(\begin{array}{cccccc|cccccc}
\mathbf{A}_{00} & \mathbf{A}_{01} & & & & & & & & & & \\
 & \mathbf{A}_{11} & \mathbf{A}_{12} & & & & & & & & & \\
 & & \ddots & \ddots & & & & & & & & \\
 & & & \mathbf{A}_{c-1,c-1} & \mathbf{A}_{c-1,c} & & & & & & & \\
 & & & & \mathbf{A}_{cc} & & \mathbf{A}_{c,c+1} & & & & & \\
\hline
 & & & & & & \mathbf{A}^*_{c+1,c+1} & \mathbf{A}_{c+1,c+2} & & & & \\
 & & & & & & & \ddots & \ddots & & & \\
 & & & & & & & & \mathbf{A}^*_{x-1,x-1} & \mathbf{A}_{x-1,c} & & \\
 & & & & & & & & & \mathbf{A}^*_{x,x} & &
\end{array}\right),$$

$$\mathbf{V}(x) = \left(\begin{array}{cccccc|cccccc}
0 & & & & & & & & & & & \\
\mathbf{A}_{10} & & & & & & & & & & & \\
 & \mathbf{A}_{21} & & & & & & & & & & \\
 & & \ddots & & & & & & & & & \\
 & & & \mathbf{A}_{c,c-1} & \mathbf{0}_{(c+1)\times(c+1)} & & & & & & & \\
\hline
 & & & & \mathbf{A}_{c+1,c} & & \overline{\mathbf{A}}^*_{c+1,c+1} & & & & & \\
 & & & & & & \mathbf{A}_{c+2,c+1} & \overline{\mathbf{A}}^*_{c+2,c+2} & & & & \\
 & & & & & & & \ddots & \ddots & & & \\
 & & & & & & & & \mathbf{A}_{x,x-1} & \overline{\mathbf{A}}^*_{xx} & &
\end{array}\right),$$

where

$$\overline{\mathbf{A}}^*_{yy} = \begin{pmatrix} 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & (y-c)\mu \end{pmatrix}, \quad y \geq c+1,$$

and the matrix $\mathbf{A}^*_{yy}$ is given by

$$\begin{pmatrix} -\lambda - y\mu & y\mu & & & & \\ & -\lambda - \nu - (y-1)\mu & (y-1)\mu & & & \\ & & \ddots & \ddots & & \\ & & & -\lambda - (c-1)\nu - (y-c+1)\mu & (y-c+1)\mu \\ & & & & -\lambda - c\nu - (y-c)\mu \end{pmatrix}$$

For $x \geq c+1$, the matrix exponential $\exp\{\mathbf{V}(x)t\}$ has the structured form

$$\left(\begin{array}{cccc|ccc}
\mathbf{M}(0;0) & & & & & & \\
\mathbf{M}(1;0) & \mathbf{M}(1;1) & & & & & \\
\vdots & \vdots & \ddots & & & & \\
\mathbf{M}(c;0) & \mathbf{M}(c;1) & \cdots & \mathbf{M}(c;c) & & & \\
\hline
\mathbf{M}(c+1;0) & \mathbf{M}(c+1;1) & \cdots & \mathbf{M}(c+1;c) & \mathbf{M}(c+1;c+1) & & \\
\vdots & \vdots & & \vdots & \vdots & \ddots & \\
\mathbf{M}(x;0) & \mathbf{M}(x;1) & \cdots & \mathbf{M}(x;c) & \mathbf{M}(x;c+1) & \cdots & \mathbf{M}(x;x)
\end{array}\right)$$

where

$$\mathbf{M}(y;y) = \begin{pmatrix} \mathbf{I}_c & \\ & e^{(y-c)\mu t} \end{pmatrix}, \quad c+1 \leq y \leq x,$$

$$\mathbf{M}(y;y') = \begin{pmatrix} \frac{t^{y-y'}}{(y-y')!}\mathbf{I}_c & \\ & ((y-c)\mu)^{y'-y}\left(e^{(y-c)\mu t} - \sum_{k=0}^{y-1-y'} \frac{((y-c)\mu t)^k}{k!}\right) \end{pmatrix}$$

$$\times \mathbf{A}_{y,y-1} \cdot \ldots \cdot \mathbf{A}_{y'+1,y'}, \quad 0 \leq y' \leq y-1, c+1 \leq y \leq x.$$

For $x \geq c+1$, the matrix exponential $\exp\{\mathbf{V}(x)t\}$ has the structured form

$$\left(\begin{array}{cccc|ccc} \mathbf{M}(0;0) & & & & & & \\ \mathbf{M}(1;0) & \mathbf{M}(1;1) & & & & & \\ \vdots & \vdots & \ddots & & & & \\ \mathbf{M}(c;0) & \mathbf{M}(c;1) & \cdots & \mathbf{M}(c;c) & & & \\ \hline \mathbf{M}(c+1;0) & \mathbf{M}(c+1;1) & \cdots & \mathbf{M}(c+1;c) & \mathbf{M}(c+1;c+1) & & \\ \vdots & \vdots & & \vdots & \vdots & \ddots & \\ \mathbf{M}(x;0) & \mathbf{M}(x;1) & \cdots & \mathbf{M}(x;c) & \mathbf{M}(x;c+1) & \cdots & \mathbf{M}(x;x) \end{array}\right)$$

where

$$\mathbf{M}(y;y) = \begin{pmatrix} \mathbf{I}_c & \\ & e^{(y-c)\mu t} \end{pmatrix}, \quad c+1 \leq y \leq x,$$

$$\mathbf{M}(y;y') = \begin{pmatrix} \frac{t^{y-y'}}{(y-y')!}\mathbf{I}_c & \\ & ((y-c)\mu)^{y'-y}\left(e^{(y-c)\mu t} - \sum_{k=0}^{y-1-y'} \frac{((y-c)\mu t)^k}{k!}\right) \end{pmatrix}$$

$$\times \mathbf{A}_{y,y-1} \cdot \ldots \cdot \mathbf{A}_{y'+1,y'}, \quad 0 \leq y' \leq y-1, c+1 \leq y \leq x.$$

For $x \geq c+1$, the matrix exponential $\exp\{\mathbf{V}(x)t\}$ has the structured form

$$\left(\begin{array}{cccc|ccc}
\mathbf{M}(0;0) & & & & & & \\
\mathbf{M}(1;0) & \mathbf{M}(1;1) & & & & & \\
\vdots & \vdots & \ddots & & & & \\
\mathbf{M}(c;0) & \mathbf{M}(c;1) & \cdots & \mathbf{M}(c;c) & & & \\
\hline
\mathbf{M}(c+1;0) & \mathbf{M}(c+1;1) & \cdots & \mathbf{M}(c+1;c) & \mathbf{M}(c+1;c+1) & & \\
\vdots & \vdots & & \vdots & \vdots & \ddots & \\
\mathbf{M}(x;0) & \mathbf{M}(x;1) & \cdots & \mathbf{M}(x;c) & \mathbf{M}(x;c+1) & \cdots & \mathbf{M}(x;x)
\end{array}\right)$$

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$$\times \mathbf{A}_{y,y-1} \cdot \ldots \cdot \mathbf{A}_{y'+1,y'}, \quad 0 \leq y' \leq y-1, c+1 \leq y \leq x.$$

For $x \geq c+1$, the matrix exponential $\exp\{\mathbf{V}(x)t\}$ has the structured form

$$\left(\begin{array}{cccc|ccc}
\mathbf{M}(0;0) & & & & & & \\
\mathbf{M}(1;0) & \mathbf{M}(1;1) & & & & & \\
\vdots & \vdots & \ddots & & & & \\
\mathbf{M}(c;0) & \mathbf{M}(c;1) & \cdots & \mathbf{M}(c;c) & & & \\
\hline
\mathbf{M}(c+1;0) & \mathbf{M}(c+1;1) & \cdots & \mathbf{M}(c+1;c) & \mathbf{M}(c+1;c+1) & & \\
\vdots & \vdots & & \vdots & \vdots & \ddots & \\
\mathbf{M}(x;0) & \mathbf{M}(x;1) & \cdots & \mathbf{M}(x;c) & \mathbf{M}(x;c+1) & \cdots & \mathbf{M}(x;x)
\end{array}\right)$$

where

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$$\times \mathbf{A}_{y,y-1} \cdot \ldots \cdot \mathbf{A}_{y'+1,y'}, \quad 0 \leq y' \leq y-1, c+1 \leq y \leq x.$$

For values $x \geq c+1$,

$$\exp\{\mathbf{U}(x)t\} = \begin{pmatrix} \exp\{\mathbf{U}(x-1)t\} & \mathbf{N}_x(t) \\ \mathbf{0}_{(c+1)\times J(x-1)} & \exp\{\mathbf{A}^*_{xx}t\} \end{pmatrix}, \quad x \geq c,$$

where $\exp\{\mathbf{A}^*_{xx}t\} = \mathbf{P}_{xx}\mathbf{T}_{xx}(t)\mathbf{P}_{xx}^{-1}$ with

$$\begin{aligned} \mathbf{N}_x(t) &= \left(\mathbf{N}_x\mathbf{T}_{xx}(t) - \exp\{\mathbf{U}(x-1)t\}\mathbf{N}_x\right)\mathbf{P}_{xx}^{-1}, \\ \mathbf{T}_{xx}(t) &= \text{diag}[e^{r(x;0)t}, e^{r(x;1)t}, ..., e^{r(x;c)t}], \\ \mathbf{P}_{xx} &= [\mathbf{v}(x;0), \mathbf{v}(x;1), ..., \mathbf{v}(x;c)], \\ \mathbf{P}_{xx}^{-1} &= \begin{pmatrix} c^{-1}(x;0)\mathbf{w}(x;0) \\ c^{-1}(x;1)\mathbf{w}(x;1) \\ \vdots \\ c^{-1}(x;x)\mathbf{w}(x;c) \end{pmatrix}, \end{aligned}$$

and $c(x;l) = \sum_{k=0}^{c}(\mathbf{w}(x;l))_k(\mathbf{v}(x;l))_k$.

Right eigenvector $\mathbf{v}(y;l)$ of $\mathbf{A}_{yy}$ with $0 \leq y \leq c$ and $\mathbf{A}^*_{yy}$ with $c+1 \leq y$, associated with the eigenvalue $r(y;l) = -(\lambda + l\nu + (y-l)\mu)$, for $0 \leq l \leq \min(y,c)$:

$$v_{l'}(y;l) = \begin{cases} \binom{y-l'}{y-l}\left(1-\frac{\nu}{\mu}\right)^{-(l-l')}, & \text{if } 0 \leq l' \leq l-1, \\ 1, & \text{if } l' = l, \\ 0, & \text{if } l+1 \leq l' \leq \min(y,c). \end{cases}$$

Left eigenvector $\mathbf{w}(y;l)$ of $\mathbf{A}_{yy}$ with $0 \leq y \leq c$ and $\mathbf{A}^*_{yy}$ with $c+1 \leq y$, associated with the eigenvalue $r(y;l) = -(\lambda + l\nu + (y-l)\mu)$, for $0 \leq l \leq \min(y,c)$:

$$w_{l'}(y;l) = \begin{cases} 0, & \text{if } 0 \leq l' \leq l-1, \\ 1, & \text{if } l' = l, \\ \binom{y-l}{y-l'}\left(\frac{\nu}{\mu}-1\right)^{-(l'-l)}, & \text{if } l+1 \leq l' \leq \min(y,c). \end{cases}$$

For values $x \geq 1$, the technical condition becomes

The eigenvalues of $\mathbf{U}(x)$ are all distinct if and only if $\nu \neq \mu$ and

$$\nu \neq \left(1 + \frac{y' - y}{l - l'}\right)\mu,$$

for every pair of integers $0 \leq y < y' \leq x$, and every integers $0 \leq l \leq \min(y, c)$ and $0 \leq l' \leq \min(y', c)$ with $l \neq l'$.

Selection of p_0: In the approximation

$$\exp\{\mathbf{T}(x)t_0\} \quad \backsimeq \quad \left(\exp\{\mathbf{U}(x)t\}\exp\{\mathbf{V}(x)t\}\right)^{p_0},$$

where $t = p_0^{-1}t_0$, we choose p_0 such that

$$\left|\left|\exp\{\mathbf{T}(x)t_0\} - \left(\exp\{\mathbf{U}(x)t\}\exp\{\mathbf{V}(x)t\}\right)^{p_0}\right|\right|_\infty \quad < \quad \varepsilon,$$

where the *maximum row sum matrix* norm is defined by $||\mathbf{W}||_\infty = \max_{1\le i\le k}\sum_{j=1}^{k}|w_{ij}|$.

Then, it is seen that

$$\left|\left|\exp\{\mathbf{T}(x)t_0\} - \left(\exp\{\mathbf{U}(x)t\}\exp\{\mathbf{V}(x)t\}\right)^{p_0}\right|\right|_\infty \quad \le \quad \frac{h(x;t_0)}{2p_0},$$

where $\mu_\infty(\cdot)$ is the logarithmic norm of a matrix and

$$h(x;t_0) \quad = \quad ||\mathbf{U}(x)\mathbf{V}(x) - \mathbf{V}(x)\mathbf{U}(x)||_\infty t_0^2 e^{(\mu_\infty(\mathbf{U}(x))+\mu_\infty(\mathbf{V}(x)))t_0}.$$

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In particular,

$$\mu_\infty(\mathbf{U}(x)) = 0, \quad x \geq 1,$$
$$\mu_\infty(\mathbf{V}(x)) = \begin{cases} x\nu, & \text{if } 1 \leq x \leq c, \\ c\nu + (x-c)\mu, & \text{if } c+1 \leq x, \end{cases}$$

For $1 \leq x \leq c$, it is readily seen that

$$||\mathbf{U}(x)\mathbf{V}(x) - \mathbf{V}(x)\mathbf{U}(x)||_\infty = \begin{cases} \nu\mu x, & \text{if } \lambda+\nu \leq \mu, \\ \nu(\lambda+\nu)x, & \text{if } \lambda+\nu > \mu. \end{cases}$$

(Similar expressions for $||\mathbf{U}(x)\mathbf{V}(x) - \mathbf{V}(x)\mathbf{U}(x)||_\infty$ in the case $c+1 \leq x$ are readily obtained.)

In the expression for $P(Z(t_0) \leq x | C(0) = i, N(0) = j)$, that is,

$$1 - \bar{\mathbf{e}}_{J(x)}(i,j) \left(\mathbf{I}_{J(x)} - \exp\{\mathbf{T}(x)t_0\} \right) \mathbf{e}_{J(x)},$$

we suggest to replace $\exp\{\mathbf{T}(x)t_0\}$ by

$$\left(\exp\{\mathbf{U}(x)t\} \exp\{\mathbf{V}(x)t\}\right)^{p_0},$$

with $t = p_0^{-1} t_0$, provided that p_0 is such that $p_0 > h(x; t_0)(2\varepsilon)^{-1}$, for a predetermined small value $\varepsilon > 0$. As a result,

$$\begin{aligned} &\left| P(Z(t_0) \leq x | C(0) = i, N(0) = j) - \hat{P}(Z(t_0) \leq x | C(0) = i, N(0) = j) \right| \\ &\leq \left|\left| \exp\{\mathbf{T}(x)t_0\} - \left(\exp\{\mathbf{U}(x)t\} \exp\{\mathbf{V}(x)t\}\right)^{p_0} \right|\right|_\infty \\ &\leq \frac{h(x; t_0)}{2p_0} < \varepsilon. \end{aligned}$$

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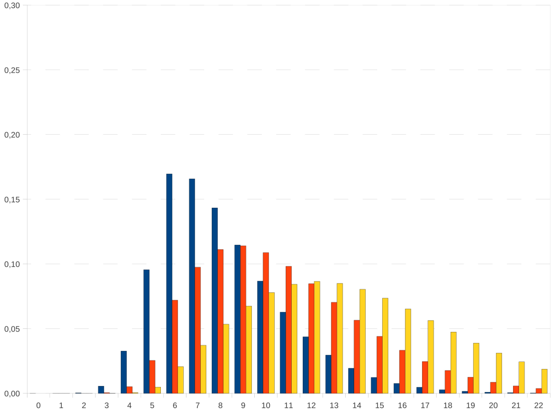
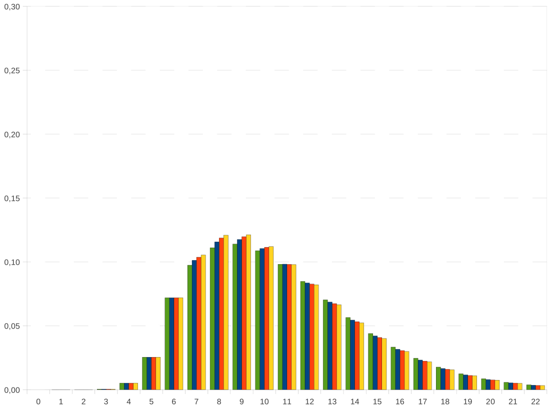
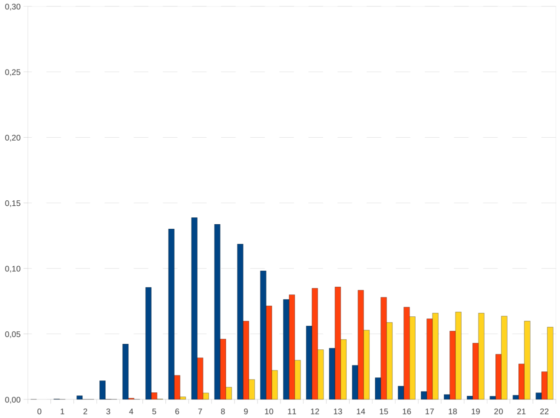


The mass function of $Z(t_0)$ versus ρ for retrial queues with $c = 6$, $\nu = 3.0\sqrt{2.0}$ and $\mu = 2.5$. Values of the traffic load: $\rho = 0.8,\ 1.0$ and 1.2 (*from left to right*); interval length: $t_0 = 1.0$; initial state: $(i, j) = (0, 0)$



The mass function of $Z(t_0)$ versus μ for retrial queues with $c = 6$, $\nu = 3.0\sqrt{2.0}$ and $\rho = 1.0$. Values of the retrial rate: $\mu = 2.5,\ 5.0,\ 7.5$ and 10.0 (*from left to right*); interval length: $t_0 = 1.0$; initial state: $(i, j) = (0, 0)$



The mass function of $Z(t_0)$ versus t_0 for retrial queues with $c = 6$, $\nu = 3.0\sqrt{2.0}$ and $\mu = 2.5$. Traffic load: $\rho = 1.5$; values of the interval length: $t_0 = 1/3$, $2/3$ and 1.0 (*from left to right*); interval length: $t_0 = 1.0$; initial state: $(i, j) = (0, 0)$

A few remarks on the time-dependent descriptor:

- It has always got a non-defective distribution, even if the LD-QBD process is not positive recurrent.
- Its probability distribution function has a matrix exponential form.
- We present simple conditions on the service rate ν and the retrial rate μ for the matrix exponential solution to be explicit or algorithmically tractable.
- We present an iterative scheme for computing the matrix exponential solution.
- A particularly appealing feature of this iterative solution based on splitting methods and eigenvalues/eigenvectors is that it allows us to obtain global error control.

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Thank you for your attention

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