# Strategic Customers in a Transportation Station: When is it Optimal to Wait?

Athanasia Manou
University of Athens, Greece

joint work with Antonis Economou and Fikri Karaesmen

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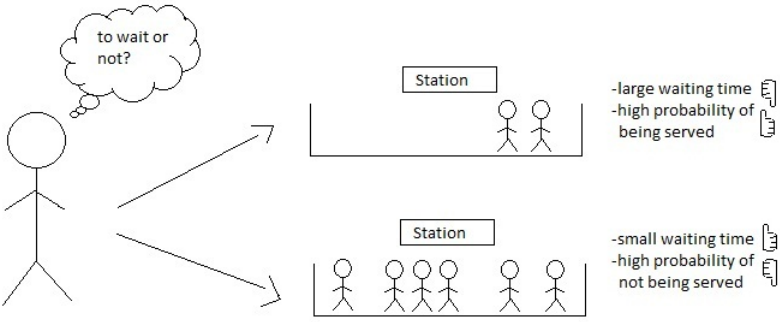
# The motivation



# The model

- Infinite waiting space (transportation station)
- Poisson($\lambda$) customers' arrival process
- 1 server (bus)
- Renewal server's visiting process $\{M(t)\}$
  Times between two visits: $X_1,\ X_2,\ X_3, \ldots \sim F(x)$
- Random server's capacities: $C_1,\ C_2,\ C_3, \ldots \sim (g_k,\ k = 1, 2, \ldots)$
- When a server with capacity $k$ visits the system:
  - serves $k$ customers instantaneously,
  - the others abandon the system

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**State description**

$$\left.\begin{array}{ll} N(t): & \text{number of customers at time } t \\ R(t): & \text{remaining time until} \\ & \text{the next server's visit at time } t \end{array}\right\} \Rightarrow \{(N(t), R(t))\} \text{ C.T.M.P.}$$

**The problem**: economic analysis of customer behavior

***join*** or ***balk***

$\Downarrow$

symmetric game among customers

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Information level:

- Unobservable case: observes nothing
- Observable case: observes $N(t)$

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Reward-Cost structure:

- Customers' reward $R$ for completing service
- Customers' waiting cost $K$ per time unit

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Decisions:

- Upon arrival a customer decides to join or to balk
- Decisions are irrevocable
- Customers' purpose: maximization of individual expected net benefit

Symmetric non-cooperative game

$S$: Set of strategies

$U(s_1, s_2)$: Payoff function of a tagged player, who follows the $s_1$ strategy, when all other players follow the $s_2$ strategy

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**Definition (Best Response)**

A strategy $s_1^*$ is said to be a best response against a strategy $s_2$, iff

$$U(s_1^*, s_2) \geq U(s_1, s_2),\ \forall s_1 \in S$$

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### Definition (Symmetric Nash Equilibrium)

A strategy $s_1^*$ is said to be a symmetric Nash equilibrium iff it is a best response against itself, i.e.

$$U(s_1^*, s_1^*) \geq U(s_1, s_1^*),\ \forall s_1 \in S$$

OBSERVABLE CASE

$\Downarrow$

customers observe $N(t)$

$\Downarrow$

strategies: $\mathbf{q} = (q_0, q_1, q_2, \ldots)$, $q_n \in [0,1]$, $n = 0, 1, \ldots$

($q_n$=probability of joining, when $N(t) = n$)

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$S_n(\mathbf{q})$: expected net benefit of a tagged customer , who finds $n$ present customers and decides to join, given that all other customers follow strategy $\mathbf{q}$.

$$S_n(\mathbf{q}) = RP[\text{service}|n, \mathbf{q}] - KE[\text{sojourn time}|n, \mathbf{q}]$$

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$$S_n(\mathbf{q}) = R\,\underbrace{P[\text{service}|n,\mathbf{q}]}_{\overset{\shortparallel}{\sum_{k=n+1}^{\infty} g_k}} - K E[\text{sojourn time}|n,\mathbf{q}]$$

$$E[\text{sojourn time}|n, \mathbf{q}]$$

$$||$$

expected residual service time at the arrival instant of a customer, who finds $n$ present customers,given that all other customers follow strategy $\mathbf{q}$

Let
$R_{\mathbf{q}}(t)$: residual service time at time $t$, when the customers follow a strategy $\mathbf{q}$,
$N_{\mathbf{q}}(t)$: number of customers in the system at time $t$, when the customers follow a strategy $\mathbf{q}$,
$\{P(t),\ t \geq 0\}$: Poisson process at rate $\lambda$,
then

$$\left\{ \begin{array}{c} \{(N_{\mathbf{q}}(u), R_{\mathbf{q}}(u)),\ 0 \leq u \leq t\}, \\ \{P(t+u) - P(t),\ u \geq 0\} \\ \text{independent} \end{array} \right\}: \text{Lack of Anticipation assumption}$$

$$\Downarrow \text{(PASTA)}$$

residual service time at the arrival instant of a customer, given that he finds $n$ present customers and that all other customers follow strategy $\mathbf{q}$

$$\shortparallel$$

residual service time at arbitrary instant, given that there are $n$ present customers in the system and that all customers follow strategy $\mathbf{q}$

Let
$\bar{n}(\mathbf{q}) = \inf\{n \geq 0 : q_i > 0 \text{ for } i < n \text{ and } q_n = 0\}$
and
$\{P_n(t),\ t \geq 0\}$: Poisson process at rate $\lambda q_n,\ n = 0, 1, \ldots, \bar{n}(\mathbf{q}) - 1$
then

$$\left\{ \begin{array}{c} \{(N_{\mathbf{q}}(u), R_{\mathbf{q}}(u)),\ 0 \leq u \leq t\}, \\ \{P_n(t+u) - P_n(t),\ u \geq 0\} \\ \text{independent} \end{array} \right\}: \text{Lack of Anticipation assumption}$$

$\Downarrow$(Conditional PASTA)

residual service time at the arrival instant of a customer, who joins, given that he finds $n$ present customers and that all customers follow strategy $\mathbf{q}$

||

residual service time at arbitrary instant, given that there are $n$ present customers in the system and that all customers follow strategy $\mathbf{q}$

For $0 \leq n < \bar{n}(\mathbf{q})$

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$R_{n,\mathbf{q}}$

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$$R_{n,\mathbf{q}}$$

## Recursive scheme for $R_{n,\mathbf{q}}$

- $R_{0,\mathbf{q}} \stackrel{d}{=} R(X), \quad \bar{n}(\mathbf{q}) = 0$
- $R_{0,\mathbf{q}} \stackrel{d}{=} (X - T_{\lambda q_0} | X \geq T_{\lambda q_0}), \quad \bar{n}(\mathbf{q}) > 0,$
  where $T_{\lambda q_0} \sim \mathrm{Exp}(\lambda q_0)$
- $R_{n,\mathbf{q}} \stackrel{d}{=} R(R_{n-1,\mathbf{q}}), \quad \bar{n}(\mathbf{q}) = n > 0$
- $R_{n,\mathbf{q}} \stackrel{d}{=} (R_{n-1,\mathbf{q}} - T_{\lambda q_n} | R_{n-1,\mathbf{q}} \geq T_{\lambda q_n}), \quad \bar{n}(\mathbf{q}) > n \geq 1,$
  where $T_{\lambda q_n} \sim \mathrm{Exp}(\lambda q_n)$

## Recursive scheme for LSTs of $R_{n,\mathbf{q}}$

**Lemma**

Let $T_1$, $T_2$ and $Y$ be independent random variables, with $T_1$ and $T_2$ being exponentially distributed with parameters $\lambda_1$ and $\lambda_2$, respectively, and $Y$ being a non-negative generally distributed random variable with LST $\tilde{F}_Y(s)$. Then we have the following formulas.

$$\Pr[Y \leq T_1] = \tilde{F}_Y(\lambda_1), \quad (1)$$

$$\Pr[Y \leq T_1 + T_2] = \frac{\lambda_2}{\lambda_2 - \lambda_1}\tilde{F}_Y(\lambda_1) + \frac{\lambda_1}{\lambda_1 - \lambda_2}\tilde{F}_Y(\lambda_2), \ \lambda_1 \neq \lambda_2, \quad (2)$$

$$\Pr[Y \leq T_1 + T_2] = \tilde{F}_Y(\lambda_1) - \lambda_1 \tilde{F}'_Y(\lambda_1), \ \lambda_1 = \lambda_2. \quad (3)$$

## Recursive scheme for LSTs of $R_{n,\mathbf{q}}$

$\tilde{F}_{n,\mathbf{q}}(s)$: the LST of $R_{n,\mathbf{q}}$
$\tilde{F}(s)$: the LST of $X$

- If $\bar{n}(\mathbf{q}) = 0$,
  $R_{0,\mathbf{q}} \stackrel{d}{=} R(X) \Rightarrow \tilde{F}_{0,\mathbf{q}}(s) = \frac{-(1-\tilde{F}(s))}{s\tilde{F}'(0)}$ (4)

- If $\bar{n}(\mathbf{q}) > 0$,
  $T_s \sim Exp(s)$
  $R_{0,\mathbf{q}} \stackrel{d}{=} (X - T_{\lambda q_0} | X \geq T_{\lambda q_0}) \Rightarrow$
  $\Pr[T_s \geq R_{0,\mathbf{q}}] = \Pr[T_s \geq (X - T_{\lambda q_0}) | (X \geq T_{\lambda q_0})] \Rightarrow$
  $\Pr[T_s \geq R_{0,\mathbf{q}}] = \frac{\Pr[T_s \geq X - T_{\lambda q_0}, X \geq T_{\lambda q_0}]}{\Pr[X \geq T_{\lambda q_0}]} \Rightarrow$
  $\Pr[T_s \geq R_{0,\mathbf{q}}] = \frac{\Pr[X \leq T_{\lambda q_0} + T_s] - \Pr[X < T_{\lambda q_0}]}{\Pr[X \geq T_{\lambda q_0}]}$ (5)
  if $s \neq \lambda q_0$, (5) $\stackrel{(1),(2)}{\Longrightarrow}$ $\tilde{F}_{0,\mathbf{q}}(s) = \frac{\lambda q_0(\tilde{F}(\lambda q_0) - \tilde{F}(s))}{(s - \lambda q_0)(1 - \tilde{F}(\lambda q_0))}$ (6)
  if $s = \lambda q_0$, (5) $\stackrel{(1),(3)}{\Longrightarrow}$ $\tilde{F}_{0,\mathbf{q}}(\lambda q_0) = \frac{-\lambda q_0 \tilde{F}'(\lambda q_0)}{1 - \tilde{F}(\lambda q_0)}$ (7)

## Recursive scheme for LSTs of $R_{n,\mathbf{q}}$

- If $\bar{n}(\mathbf{q}) = n \geq 1$,
  $R_{n,\mathbf{q}} \stackrel{d}{=} R(R_{n-1,\mathbf{q}}) \Rightarrow \tilde{F}_{n,\mathbf{q}}(s) = \frac{-(1-\tilde{F}_{n-1,\mathbf{q}}(s))}{s\tilde{F}'_{n-1,\mathbf{q}}(0)}$ (8)
- If $\bar{n}(\mathbf{q}) > n \geq 1$ and $s \neq \lambda q_n$,
  $R_{n,\mathbf{q}} \stackrel{d}{=} (R_{n-1,\mathbf{q}} - T_{\lambda q_n} | R_{n-1,\mathbf{q}} \geq T_{\lambda q_n}) \Rightarrow$
  $\tilde{F}_{n,\mathbf{q}}(s) = \frac{\lambda q_n (\tilde{F}_{n-1,\mathbf{q}}(\lambda q_n) - \tilde{F}_{n-1,\mathbf{q}}(s))}{(s-\lambda q_n)(1-\tilde{F}_{n-1,\mathbf{q}}(\lambda q_n))}$ (9)
- If $\bar{n}(\mathbf{q}) > n \geq 1$ and $s = \lambda q_n$,
  $R_{n,\mathbf{q}} \stackrel{d}{=} (R_{n-1,\mathbf{q}} - T_{\lambda q_n} | R_{n-1,\mathbf{q}} \geq T_{\lambda q_n}) \Rightarrow$
  $\tilde{F}_{n,\mathbf{q}}(\lambda q_n) = \frac{-\lambda q_n \tilde{F}'_{n-1,\mathbf{q}}(\lambda q_n)}{1-\tilde{F}_{n-1,\mathbf{q}}(\lambda q_n)}$ (10)

$\tilde{F}_{n,\mathbf{q}}(s)$ depends on $\mathbf{q}$ only through $\mathbf{q}_n = (q_0, q_1, q_2, \ldots, q_n)$. So, we can write
$\tilde{F}_{n,\mathbf{q}}(s) = \tilde{F}_{n,\mathbf{q}_n}(s)$
$\tilde{F}_{n,\mathbf{q}_n}(s)$ is continuous in $q_n$

# Recursive scheme for $E[R_{n,\mathbf{q}}]$

## Corollary (Expected sojourn times)

Consider the observable model of a transportation station, where customers join the system according to a strategy $\mathbf{q} = (q_0, q_1, q_2, \ldots)$. For the expected conditional residual service times, $E[R_{n,\mathbf{q}_n}]$, we have the following recursive scheme

$$E[R_{n,\mathbf{q}_n}] = \frac{E[R_{n-1,\mathbf{q}_{n-1}}]}{1-\tilde{F}_{n-1,\mathbf{q}_{n-1}}(\lambda q_n)} - \frac{1}{\lambda q_n},\ q_i \neq 0,\ i = 0,1,\ldots,n,\ n \geq 1,$$

$$E[R_{n,\mathbf{q}_n}] = \frac{E[R^2_{n-1,\mathbf{q}_{n-1}}]}{2E[R_{n-1,\mathbf{q}_{n-1}}]},\ q_i \neq 0,\ i = 0,1,\ldots,n-1,\ q_n = 0,\ n \geq 1,$$

with initial condition

$$E[R_{0,q_0}] = \frac{E[X]}{1-\tilde{F}(\lambda q_0)} - \frac{1}{\lambda q_0},\ q_0 \neq 0$$

$$E[R_{0,q_0}] = \frac{E[X^2]}{2E[X]},\ q_0 = 0.$$

## Proposition (Expected net benefit)

Consider the observable model of a transportation station, where the customers join the system according to a strategy $\mathbf{q} = (q_0, q_1, q_2, \ldots)$. Then, the expected net benefit $S_n(\mathbf{q})$ of an arriving customer, who finds n present customers in the system and decides to join, is given by the formulas

$$\begin{aligned}
S_n(\mathbf{q}) &= R\sum_{k=n+1}^{\infty} g_k - K\left[\frac{E[R_{n-1,\mathbf{q}_{n-1}}]}{1-\tilde{F}_{n-1,\mathbf{q}_{n-1}}(\lambda q_n)} - \frac{1}{\lambda q_n}\right],\ q_i \neq 0, \\
&\qquad i = 0,1,\ldots,n,\ n \geq 1, \\
S_n(\mathbf{q}) &= R\sum_{k=n+1}^{\infty} g_k - K\frac{E[(R_{n-1,\mathbf{q}_{n-1}})^2]}{2E[R_{n-1,\mathbf{q}_{n-1}}]},\ q_i \neq 0,\ i = 0,1,\ldots,n-1, \\
&\qquad q_n = 0,\ n \geq 1, \\
S_0(\mathbf{q}) &= R - K\left[\frac{E[X]}{1-\tilde{F}(\lambda q_0)} - \frac{1}{\lambda q_0}\right],\ q_0 \neq 0, \\
S_0(\mathbf{q}) &= R - K\frac{E[X^2]}{2E[X]},\ q_0 = 0.
\end{aligned}$$

# Equilibrium strategies

Recursive scheme for the computation of equilibrium probabilities:

- Computation of $q_0^e$

**Theorem (Equilibrium probability $q_0^e$)**

Consider the observable model of a transportation station. Then, an equilibrium probability $q_0^e$ for joining when finding the system empty exists. Specifically, we have the following comprehensive (but not necessarily mutually exclusive) cases:

Case I: $\frac{R}{K} \leq \frac{E[X^2]}{2E[X]}$.
Then, $q_0^e = 0$.

Case II: $\frac{R}{K} \geq \frac{E[X]}{1-\tilde{F}(\lambda)} - \frac{1}{\lambda}$ .
Then, $q_0^e = 1$.

Case III: $\frac{E[X^2]}{2E[X]} < \frac{R}{K} < \frac{E[X]}{1-\tilde{F}(\lambda)} - \frac{1}{\lambda}$.
Then, there exists a $q_0'$ such that $0 < q_0' < 1$ and $\frac{E[X]}{1-\tilde{F}(\lambda q_0')} - \frac{1}{\lambda q_0'} = \frac{R}{K}$. The equilibrium joining probability is $q_0^e = q_0'$.

- Computation of $q_n^e$, given the $\mathbf{q}_{n-1}^e$

**Theorem (Equilibrium probability $q_n^e$)**

Consider the observable model of a transportation station. Then, assuming that an equilibrium joining probability vector $\mathbf{q}_{n-1}^e$ is known, an equilibrium probability $q_n^e$ for joining when finding $n$ present customers in the system exists. Specifically, we have the following cases:

Case I: $\frac{R\sum_{k=n+1}^{\infty} g_k}{K} \leq \frac{E[(R_{n-1,\mathbf{q}_{n-1}^e})^2]}{2E[R_{n-1,\mathbf{q}_{n-1}^e}]}$.
Then, $q_n^e = 0$.

Case II: $\frac{R\sum_{k=n+1}^{\infty} g_k}{K} \geq \frac{E[R_{n-1,\mathbf{q}_{n-1}^e}]}{1-\tilde{F}_{n-1,\mathbf{q}_{n-1}^e}(\lambda)} - \frac{1}{\lambda}$ .
Then, $q_n^e = 1$.

Case III: $\frac{E[(R_{n-1,\mathbf{q}_{n-1}^e})^2]}{2E[R_{n-1,\mathbf{q}_{n-1}^e}]} < \frac{R\sum_{k=n+1}^{\infty} g_k}{K} < \frac{E[R_{n-1,\mathbf{q}_{n-1}^e}]}{1-\tilde{F}_{n-1,\mathbf{q}_{n-1}^e}(\lambda)} - \frac{1}{\lambda}$.
Then, there exists a $q_n'$ such that $0 < q_n' < 1$ and $\frac{E[R_{n-1,\mathbf{q}_{n-1}^e}]}{1-\tilde{F}n-1,\mathbf{q}_{n-1}^e(\lambda q_n')} - \frac{1}{\lambda q_n'} = \frac{R\sum_{k=n+1}^{\infty} g_k}{K}$. The equilibrium joining probability is $q_n^e = q_n'$.

# BIBLIOGRAPHY

1. Burnetas, A. and Economou, A. (2007) Equilibrium customer strategies in a single server Markovian queue with setup times. *Queueing Systems* **56**, 213-228.
2. Economou, A. and Kanta, S. (2008b) Equilibrium balking strategies in the observable single-server queue with breakdowns and repairs. *Operations Research Letters* **36**, 696-699.
3. Hassin, R. (2007) Information and uncertainty in a queuing system. *Probability in the Engineering and Informational Sciences* **21**, 361-380.
4. Hassin, R. and Haviv, M. (1997) Equilibrium threshold strategies: the case of queues with priorities. *Operations Research* **45**, 966-973.
5. Hassin, R. and Haviv, M. (2003) *To Queue or Not to Queue: Equilibrium Behavior in Queueing Systems.* Kluwer Academic Publishers, Boston.

Thank you!