UNIVERSITY OF TWENTE.

The $PH/PH/1$ multi-threshold queue

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joint work with Richard J. Boucherie (UT) and
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Overview

Introduction

$PH/PH/1$ multi-threshold queue

LDQBD

Generalisations

Conclusion

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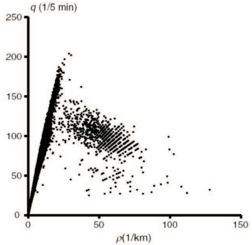
Introduction

PhD-student since November 2010.

Working on a queueing network able to mimic behaviour of highway traffic.

Traffic Behaviour

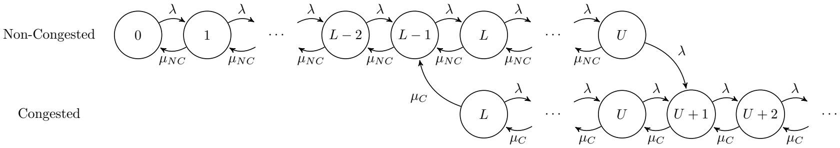
Capacity Drop - The transition from non-congested to congested traffic when density reaches an *upper* critical density. The density needs to decrease to a *lower* critical density to resolve congestion.



Shockwaves - The propagation of the capacity drop along traffic.

$M/M/1$ threshold queue

Capacity Drop - The transition from non-congested to congested traffic when density reaches a critical density. The density needs to decrease to a lower critical density to resolve congestion.



Overview

Model Description

- $PH/PH/1$ queue with **stage**-dependent service and arrival processes.
- In stage s, $PH(\mathbf{\Lambda}_s, \lambda_s)$ and $PH(\boldsymbol{M}_s, \mu_s)$.

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- Each stage s has a *lower* threshold L_s and an *upper* threshold U_s, but $L_0 = 0$ and $U_S = \infty$.

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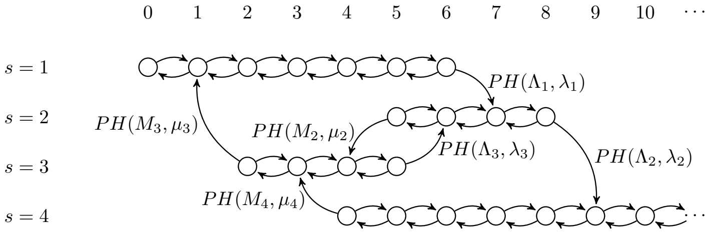
- In stage s, $PH(\mathbf{\Lambda}_s, \lambda_s)$ and $PH(\boldsymbol{M}_s, \mu_s)$.
- S-stages controlled by a threshold policy.
- Each stage s has a *lower* threshold L_s and an *upper* threshold U_s, but $L_0 = 0$ and $U_S = \infty$.
- Arrival in $U_s : (U_s, s) \to (U_{s+1}, u_s)$.
  Departure in $L_s : (L_s, s) \to (L_{s-1}, l_s)$.

Model Description



	L_s	l_s	u_s	U_s
$s=1$	0	–	2	6
$s=2$	5	3	4	8
$s=3$	2	1	2	5
$s=4$	4	3	–	∞

Overview

Level Dependent Quasi-Birth-and-Death process.

We model this queue as a Level Dependent Quasi-Birth-and-Death process.

Level: Number of customers.
Phase: Combination of **stage** and **state** of the *PH*-distributions.

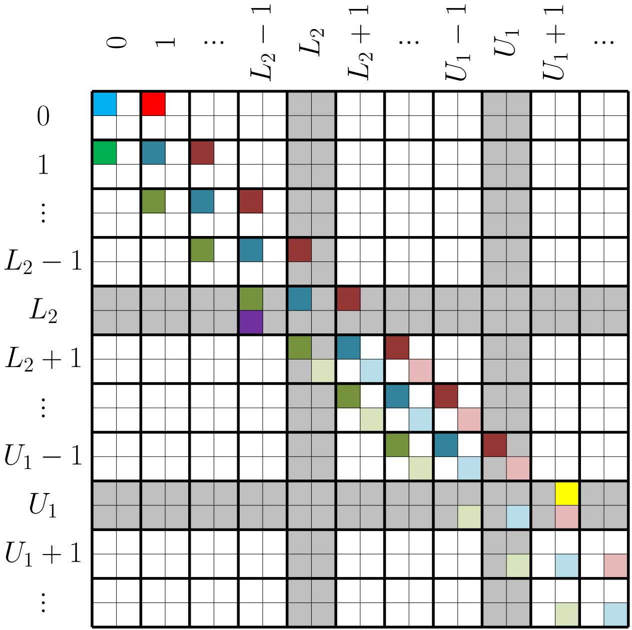
Generator

$$\boldsymbol{Q} = \begin{bmatrix} \boldsymbol{L}^{(0)} & \boldsymbol{F}^{(0)} & 0 & \cdots & & & \\ \boldsymbol{B}^{(1)} & \boldsymbol{L}^{(1)} & \boldsymbol{F}^{(1)} & \ddots & & & \\ 0 & \boldsymbol{B}^{(2)} & \boldsymbol{L}^{(2)} & \ddots & & & \\ \vdots & \ddots & \ddots & \ddots & \boldsymbol{F}^{(i-1)} & & \\ & & & \boldsymbol{B}^{(i)} & \boldsymbol{L}^{(i)} & \ddots & \\ & & & & \ddots & \ddots & \end{bmatrix}.$$

$$\boldsymbol{F}^{(i)}_{(s,j)} = \begin{cases} \boldsymbol{\Lambda}^0_s \otimes \lambda_s \otimes \boldsymbol{I}_{q_s} & \text{if } j = s \text{ and } L_s \leq i < U_s, \\ \boldsymbol{\Lambda}^0_s \otimes \boldsymbol{e}_{q_s} \otimes \lambda_{u_s} \otimes \mu_{u_s} & \text{if } j = u_s \text{ and } i = U_s, \\ 0 & \text{otherwise.} \end{cases}$$

$$\boldsymbol{L}^{(i)}_{(s,j)} = \begin{cases} \boldsymbol{\Lambda}_s \otimes \boldsymbol{I}_{q_s} + \boldsymbol{I}_{p_s} \otimes \boldsymbol{M}_s & \text{if } j = s, i > 0 \text{ and } L_s \leq i \leq U_s, \\ \boldsymbol{\Lambda}_s \otimes \boldsymbol{I}_{q_s} & \text{if } j = s = 1 \text{ and } i = 0, \\ 0 & \text{otherwise.} \end{cases}$$

$$\boldsymbol{B}^{(i)}_{(s,j)} = \begin{cases} \boldsymbol{I}_{p_s} \otimes \boldsymbol{M}^0_s \otimes \mu_s & \text{if } j = s \text{ and } L_s < i \leq U_s, \\ \boldsymbol{e}_{p_s} \otimes \boldsymbol{M}^0_s \otimes \lambda_{l_s} \otimes \mu_{l_s} & \text{if } j = l_s \text{ and } i = L_s, \\ 0 & \text{otherwise.} \end{cases}$$


0
1
...
L_2-1
L_2
L_2+1
...
U_1-1
U_1
U_1+1
...

Stationary Queue Length probabilities

$$\boldsymbol{\pi_i} = \boldsymbol{\pi_0} \prod_{n=0}^{i-1} \boldsymbol{R}^{(n)}$$

$$\boldsymbol{\pi_0} \left(\boldsymbol{L}^{(0)} + \boldsymbol{R}^{(0)} \boldsymbol{B}^{(1)} \right) = 0$$

$$\boldsymbol{\pi_0} \left(\sum_{i=0}^{\infty} \prod_{n=0}^{i-1} \boldsymbol{R}^{(n)} \right) \boldsymbol{e} = 1$$

with $\boldsymbol{R}^{(i)}$ the minimal nonnegative solution to

$$\boldsymbol{F}^{(i)} + \boldsymbol{R}^{(i)} \boldsymbol{L}^{(i+1)} + \boldsymbol{R}^{(i)} \boldsymbol{R}^{(i+1)} \boldsymbol{B}^{(i+2)} = \boldsymbol{0}, \quad i \geq 0$$

Stationary Queue Length probabilities

$$\mathbf{0} = \boldsymbol{F}^{(i)} + \boldsymbol{R}^{(i)}\boldsymbol{L}^{(i+1)} + \boldsymbol{R}^{(i)} + \boldsymbol{R}^{(i+1)}\boldsymbol{B}^{(i+2)}, \quad i \geq 0$$

$$\boldsymbol{R}^{(i)} = -\boldsymbol{F}^{(i)}\left(\boldsymbol{L}^{(i+1)} + \boldsymbol{R}^{(i+1)}\boldsymbol{B}^{(i+2)}\right)^{-1}$$

- Find $R^{(K)}$, K large enough.
- Does the inverse exist?

- System is level independent above level

$$\max_{s, U_s < \infty} \{U_s\} + 1.$$

$\boldsymbol{R}^{(K)}$ is the minimal nonnegative solution to

$$\boldsymbol{F} + \boldsymbol{R}\boldsymbol{L} + \boldsymbol{R}^2\boldsymbol{B} = 0$$

- Suppose $S = 3$, then in the tail:

$$\boldsymbol{L} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & L \end{bmatrix} \quad \boldsymbol{B} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & B \end{bmatrix} \quad \boldsymbol{R} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & R \end{bmatrix}$$

and $[\boldsymbol{L} + \boldsymbol{R}\boldsymbol{B}]^{-1}$ does not exist.

Decomposition Method

Solve

$$\boldsymbol{F}^{(i)} + \boldsymbol{R}^{(i)}\boldsymbol{L}^{(i+1)} + \boldsymbol{R}^{(i)}\boldsymbol{R}^{(i+1)}\boldsymbol{B}^{(i+2)} = \boldsymbol{0}, \quad i \geq 0$$

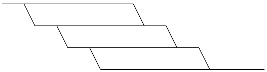
or for $j, k = 1, \ldots, S$

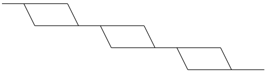
$$\boldsymbol{F}^{(i)}_{(j,k)} + \sum_{a=1}^{S} \boldsymbol{R}^{(i)}_{(j,a)}\boldsymbol{L}^{(i+1)}_{(a,k)} + \sum_{a=1}^{S}\sum_{b=1}^{S} \boldsymbol{R}^{(i)}_{(j,a)}\boldsymbol{R}^{(i+1)}_{(a,b)}\boldsymbol{B}^{(i+2)}_{(b,k)} = \boldsymbol{0}, \quad i \geq 0$$

Most cases result in solving these linear equations.

Explicit Results

If: $\boldsymbol{F}^{(i)}$ is upper triangular, $\boldsymbol{B}^{(i)}$ is lower triangular and $\boldsymbol{B}^{(i)}_{(j,k)} = \mathbf{0}$ for $j - k > 1$, **then** $\boldsymbol{R}^{(i)}$ is upper triangular





Explicit Results

If: $\boldsymbol{F}^{(i)}$ is upper triangular, $\boldsymbol{B}^{(i)}$ is lower triangular and $\boldsymbol{B}^{(i)}_{(j,k)} = \mathbf{0}$ for $j - k > 1$, **then** $\boldsymbol{R}^{(i)}$ is upper triangular and

$$\boldsymbol{F}^{(i)}_{(j,k)} + \sum_{a=1}^{S} \boldsymbol{R}^{(i)}_{(j,a)} \boldsymbol{L}^{(i+1)}_{(a,k)} + \sum_{a=1}^{S} \sum_{b=1}^{S} \boldsymbol{R}^{(i)}_{(j,a)} \boldsymbol{R}^{(i+1)}_{(a,b)} \boldsymbol{B}^{(i+2)}_{(b,k)} = \mathbf{0}, \quad i \geq 0$$

can be simplified to find

$$\boldsymbol{R}^{(i)}_{(j,j)} = f(\boldsymbol{R}^{(i+1)})$$
$$\boldsymbol{R}^{(i)}_{(j,k)} = g(\boldsymbol{R}^{(i+1)}, \boldsymbol{R}^{(i)}_{(j,j)}, \ldots, \boldsymbol{R}^{(i)}_{(j,k-1)})$$

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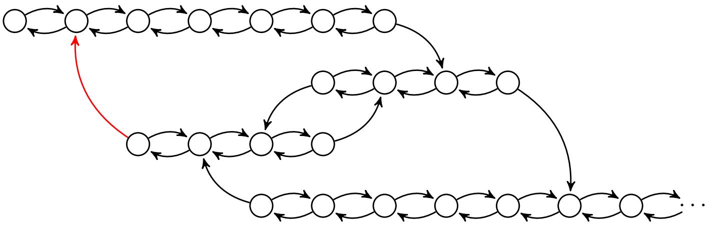
Generalisations

- Multiple zeros
- Multiple tails
- Stochastic stage changes

Multiple zeros



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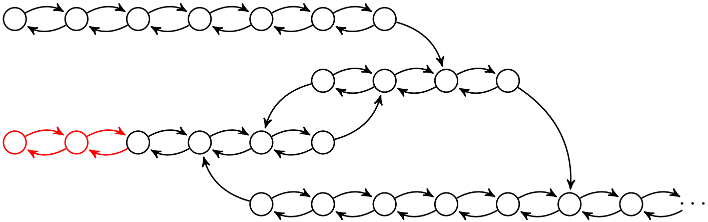
Multiple zeros



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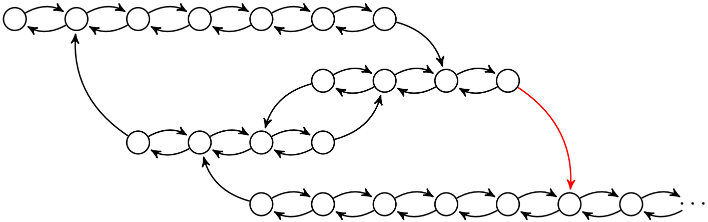
$$\boldsymbol{L}^{(i)}_{(s,j)} = \begin{cases} \boldsymbol{\Lambda}_s \otimes \boldsymbol{I}_{q_s} + \boldsymbol{I}_{p_s} \otimes \boldsymbol{M}_s & \text{if } j = s, i > 0 \text{ and } L_s \leq i \leq U_s, \\ \boldsymbol{\Lambda}_s \otimes \boldsymbol{I}_{q_s} & \text{if } j = s = 1, 3 \text{ and } i = 0, \\ 0 & \text{otherwise.} \end{cases}$$

$$\boldsymbol{B}^{(i)}_{(s,j)} = \begin{cases} \boldsymbol{I}_{p_s} \otimes \boldsymbol{M}^0_s \otimes \mu_s & \text{if } j = s \text{ and } L_s < i \leq U_s, \\ \boldsymbol{e}_{p_s} \otimes \boldsymbol{M}^0_s \otimes \lambda_{l_s} \otimes \mu_{l_s} & \text{if } j = l_s \text{ and } i = L_s, \\ 0 & \text{otherwise.} \end{cases}$$

Multiple tails



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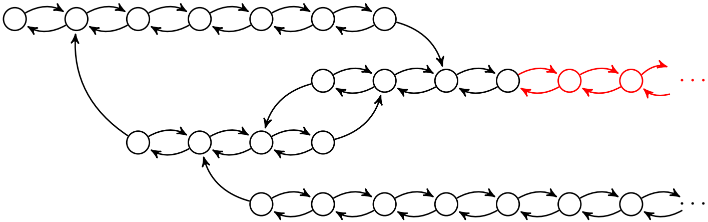

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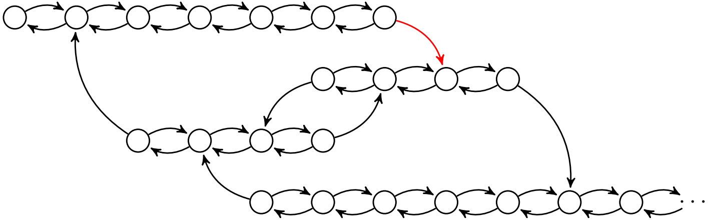
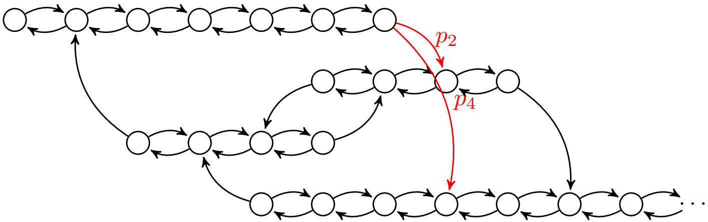
Stochastic stage changes



Stochastic stage changes



$$\boldsymbol{F}_{(s,j)}^{(i)} = \begin{cases} \boldsymbol{\Lambda}_s^0 \otimes \lambda_s \otimes \boldsymbol{I}_{q_s} & \text{if } j = s \text{ and } L_s \leq i < U_s, \\ p_{u_s} \cdot \boldsymbol{\Lambda}_s^0 \otimes \boldsymbol{e}_{q_s} \otimes \lambda_{u_s} \otimes \mu_{u_s} & \text{if } j = u_s \text{ and } i = U_s, \\ 0 & \text{otherwise.} \end{cases}$$

$$\boldsymbol{L}_{(s,j)}^{(i)} = \begin{cases} \boldsymbol{\Lambda}_s \otimes \boldsymbol{I}_{q_s} + \boldsymbol{I}_{p_s} \otimes \boldsymbol{M}_s & \text{if } j = s, i > 0 \text{ and } L_s \leq i \leq U_s, \\ \boldsymbol{\Lambda}_s \otimes \boldsymbol{I}_{q_s} & \text{if } j = s = 1 \text{ and } i = 0, \\ 0 & \text{otherwise.} \end{cases}$$

$$\boldsymbol{B}_{(s,j)}^{(i)} = \begin{cases} \boldsymbol{I}_{p_s} \otimes \boldsymbol{M}_s^0 \otimes \mu_s & \text{if } j = s \text{ and } L_s < i \leq U_s, \\ \boldsymbol{e}_{p_s} \otimes \boldsymbol{M}_s^0 \otimes \lambda_{l_s} \otimes \mu_{l_s} & \text{if } j = l_s \text{ and } i = L_s, \\ 0 & \text{otherwise.} \end{cases}$$

Overview

Conclusions and Further Research

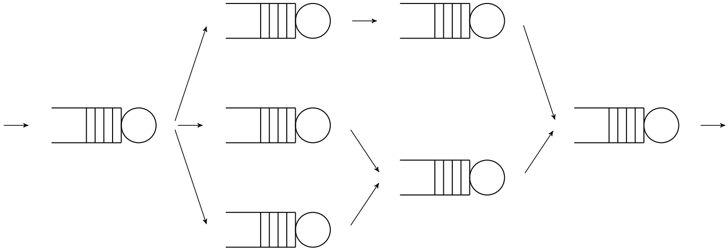
For now:

- Decomposition Method for the $PH/PH/1$ multi-threshold queue.

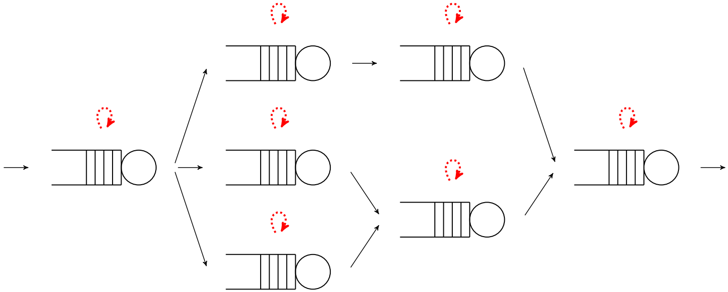
The next steps:

- Analyse (tandem) networks of $PH/PH/1$ threshold queues.
- Analyse (tandem) networks of $PH/PH/1$ **feedback** threshold queues.

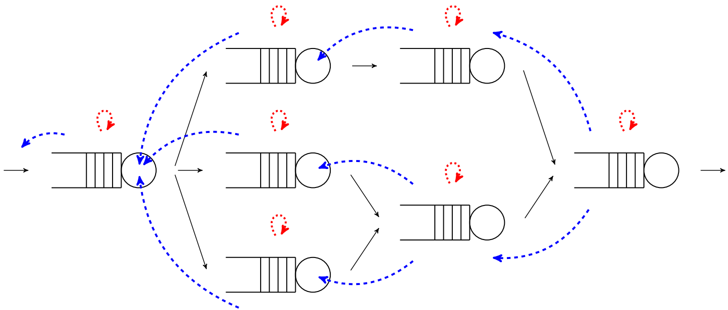
Networks



Networks with thresholds



Networks with feedback thresholds



Questions?

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