# Analysis of queueing models with multiple waiting lines: complex-function methods

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# 1. Introduction

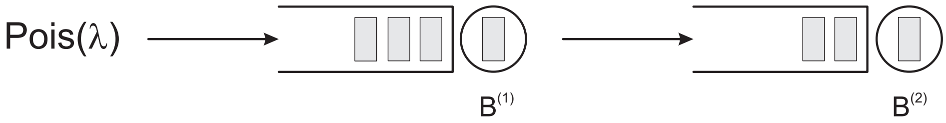
In queueing theory, there is a sharp division between easy and hard.

Single server queue: much is known.

Two-queue models: various techniques for special cases
(like the compensation approach and the boundary value technique).

$N$-queue models with $N > 2$: exact analysis in exceptional cases
(like product-form networks and polling systems).

Example: tandem queue.




$B^{(i)} \sim \exp(\mu_i)$, $i = 1, 2$:
*product-form network*, independence of queue lengths $X_i$:

$$\mathbb{P}(X_1 = n_1, X_2 = n_2) = \prod_{i=1}^{2} (1 - \rho_i)\rho_i^{n_i},$$

with $\rho_i := \lambda \mathbb{E} B^{(i)} = \frac{\lambda}{\mu_i} < 1$.

$B^{(2)}$ general: queue lengths still independent, product form.

$B^{(1)}$ general, $B^{(2)} \sim \exp$: boundary value problem
Blanc, Iasnogorodski and Nain (1988).

$B^{(1)}$ and $B^{(2)}$ general: hopeless?

Starting-point often: balance equations for queue lengths.
Two $M/M/1$ queues in series:

$$(\lambda+\mu_1+\mu_2)p(m,n) = \lambda p(m-1,n)I_{m\geq 1}+\mu_1 p(m+1,n-1)I_{n\geq 1}+\mu_2 p(m,n+1).$$

Together with $\sum\sum p(m,n) = 1$, this contains *all* the information.

Now take generating functions, with $P(x,y) := \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} x^m y^n p(m,n)$:

$$K(x,y)P(x,y) = H(x,y)P(x,0) + V(x,y)P(0,y) + O(x,y)P(0,0).$$

In addition, $P(1,1) = 1$, and $P(x,y)$ is analytic for $|x| \leq 1$, $|y| \leq 1$.
Together, this contains *all* the information needed to determine $P(x,y)$.
For the tandem queue, prove that

$$P(x,y) = \frac{1-\rho_1}{1-\rho_1 x}\frac{1-\rho_2}{1-\rho_2 y}.$$

$$K(x,y)P(x,y) = H(x,y)P(x,0) + V(x,y)P(0,y) + O(x,y)P(0,0).$$

Approach:
Consider all zeros $(\hat{x}, \hat{y})$ of kernel $K(x,y)$ with $|\hat{x}| \leq 1$, $|\hat{y}| \leq 1$.

Embarrassment of choice!
What is a good, systematic choice?

The uniformization method determines a suitable set of zero pairs $(\hat{x}(z), \hat{y}(z))$, $z \in L$, for some smooth closed contour $L$ which is itself determined;
and $|\hat{x}(z)| \leq 1$, $|\hat{y}(z)| \leq 1$,
and $\hat{x}(z)$ analytic in $L^+$, $\hat{y}(z)$ analytic in $L^-$.

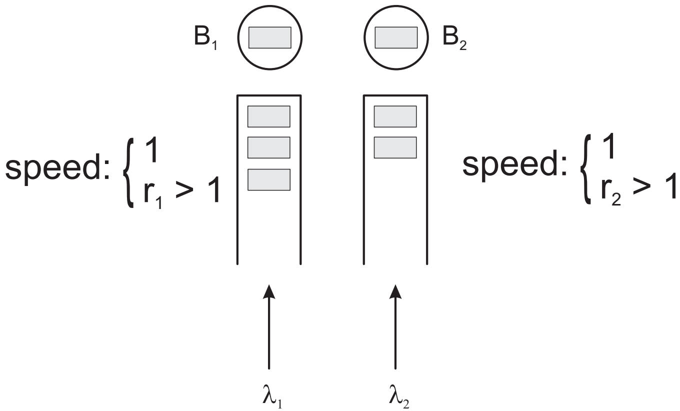
Then necessarily, for all $z \in L$:

$$H(\hat{x}(z), \hat{y}(z))P(\hat{x}(z), 0) = -V(\hat{x}(z), \hat{y}(z))P(0, \hat{y}(z)) - O(\hat{x}(z), \hat{y}(z))P(0, 0).$$

analytic in $L^+$ analytic in $L^-$

Now try to formulate a boundary value problem,
to solve $P(\hat{x}(z), 0)$, $z \in L^+$, and $P(0, \hat{y}(z))$, $z \in L^-$.

Some historical remarks:
Breakthrough for *two*-dimensional queues: Fayolle and Iasnogorodski (1979)
Joint queue length distribution for two coupled processors with exponential service times: reduction to a Riemann-Hilbert problem.




Further development of the theory: Cohen and Boxma (1983), Cohen (1992).
General service times, reduction to a Riemann or Riemann-Hilbert or Wiener-Hopf problem or Fredholm singular integral equation.

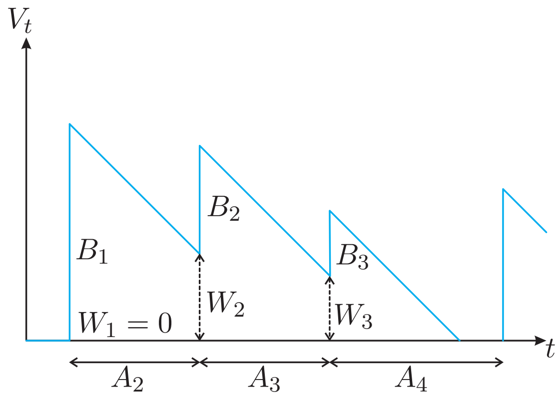
## Outline of the talk

1. Introduction
2. The $GI/GI/1$ queue: The Wiener-Hopf boundary value technique
3. Two coupled processors: The Wiener-Hopf boundary value technique
4. Closing remarks: relation between the compensation approach and complex function techniques

## 2. The $GI/GI/1$ queue: The Wiener-Hopf boundary value technique

Norbert Wiener and Eberhard Hopf (1931):
radiation of stars $\Longrightarrow$ integral equation $\Longrightarrow$ boundary value problem for analytic functions

i.i.d. interarrival times $A_1, A_2, \ldots$; LST $\alpha(s) = \mathbb{E}[\mathrm{e}^{-sA}]$.
i.i.d. service times $B_1, B_2, \ldots$; LST $\beta(s) = \mathbb{E}[\mathrm{e}^{-sB}]$. Load $\rho := \frac{\mathbb{E}B}{\mathbb{E}A} < 1$.




waiting time $W_{n+1}$: $W_{n+1} = \max(0, W_n + B_n - A_{n+1})$, $n = 1, 2, \ldots$

Let $|r| \leq 1$ and

$$\Phi(r,s) := \sum_{n=1}^{\infty} r^n \mathbb{E}[\mathrm{e}^{-sW_n} | W_1 = 0], \quad \mathrm{Re}\, s \geq 0,$$

$$\Psi(r,s) := \sum_{n=1}^{\infty} r^n \mathbb{E}[\mathrm{e}^{-s\min(0,W_n+B_n-A_{n+1})} | W_1 = 0], \quad \mathrm{Re}\, s \leq 0.$$

Use the identity

$$\mathrm{e}^{-sW_{n+1}} = \mathrm{e}^{-s(W_n+B_n-A_{n+1})} - \mathrm{e}^{-s\min(0,W_n+B_n-A_{n+1})} + 1,$$

to get, for $|r| < 1$ and $\mathrm{Re}\, s = 0$:

$$\Phi(r,s)[1 - r\alpha(-s)\beta(s)] = \frac{r}{1-r} - r\Psi(r,s).$$

## Wiener-Hopf boundary value problem:

Find $\Phi(r,s)$ and $\Psi(r,s)$, such that,
(i) for $|r| < 1$ and $\mathrm{Re}\ s = 0$:

$$\Phi(r,s)[1 - r\alpha(-s)\beta(s)] = \frac{r}{1-r} - r\Psi(r,s).$$

(ii) $\Phi(r,s)$ is analytic for $\mathrm{Re}\ s > 0$, continuous for $\mathrm{Re}\ s \geq 0$,
bounded (by $\frac{r}{1-r}$) for $\mathrm{Re}\ \ s \geq 0$.

(iii) $\Psi(r,s)$ is analytic for $\mathrm{Re}\ s < 0$, continuous for $\mathrm{Re}\ s \leq 0$,
bounded (by $\frac{r}{1-r}$) for $\mathrm{Re}\ \ s \leq 0$.

$$\Phi(r,s)[1 - r\alpha(-s)\beta(s)] = \frac{r}{1-r} - r\Psi(r,s).$$

Wiener-Hopf technique: rewrite

$$\frac{1}{1 - r\alpha(-s)\beta(s)} = \phi_+(r,s)\phi_0(r)\phi_-(r,s),$$

with $\phi_+(r,s)$ ($\phi_-(r,s)$) analytic and non-zero in $\mathrm{Re}\, s > 0$ $(< 0)$.

Then

$$\frac{\Phi(r,s)}{\phi_+(r,s)} = [\frac{1}{1-r} - \Psi(r,s)]r\phi_0(r)\phi_-(r,s).$$

LHS: analytic for $\mathrm{Re}\, s > 0$. RHS: analytic for $\mathrm{Re}\, s < 0$.

Let $Z(r,s) := LHS$ for $\mathrm{Re}\, s \geq 0$ and $Z(r,s) := RHS$ for $\mathrm{Re}\, s \leq 0$.
Then $Z(r,s)$ is analytic, and also bounded, in the whole $s$-plane.

Liouville's theorem: a function of $s$ that is analytic and bounded in the whole $s$-plane is a *constant*. So here: $Z(r, s) = C(r)$.

Consequence:

$$\frac{\Phi(r, s)}{\phi_+(r, s)} = C(r), \quad \text{Re } s \geq 0;$$

$$[\frac{1}{1-r} - \Psi(r, s)] r \phi_0(r) \phi_-(r, s) = C(r), \quad \text{Re } s \leq 0.$$

Using $\Phi(r, 0) = \frac{r}{1-r}$ one obtains

$$\Phi(r, s) = C(r)\phi_+(r, s) = \frac{r}{1-r} \frac{\phi_+(r, s)}{\phi_+(r, 0)}, \quad \text{Re } s \geq 0.$$

Similarly $\Psi(r, s)$.

The hard part: how to factorize

$$\frac{1}{1-r\alpha(-s)\beta(s)} = \phi_+(r,s)\phi_0(r)\phi_-(r,s).$$

Spitzer's identity (with $S_n := \sum_{k=1}^{n}(B_k - A_{k+1})$):

$$\frac{1}{1-r\alpha(-s)\beta(s)} = \exp[-\ln(1-r\alpha(-s)\beta(s))]$$

$$= \exp[\sum_{n=1}^{\infty}\frac{r^n}{n}\{\alpha(-s)\beta(s)\}^n] = \exp[\sum_{n=1}^{\infty}\frac{r^n}{n}\mathbb{E}[e^{-sS_n}]]$$

$$= \exp[\sum_{n=1}^{\infty}\frac{r^n}{n}\mathbb{E}[e^{-sS_n}I(S_n>0)]]\exp[\sum_{n=1}^{\infty}\frac{r^n}{n}\mathbb{E}[e^{-sS_n}I(S_n<0)]]$$

$$\times \exp[\sum_{n=1}^{\infty}\frac{r^n}{n}\mathbb{E}[e^{-sS_n}I(S_n=0)]].$$

Special case: $M/G/1$; $\alpha(s) = \frac{\lambda}{\lambda+s}$.

$$\Phi(r,s)\frac{\lambda - s - r\lambda\beta(s)}{\lambda - s} = \frac{r}{1-r} - r\Psi(r,s), \quad |r| < 1, \ \ \mathrm{Re}\, s = 0.$$

LHS: one pole $s = \lambda > 0$ and one zero $s = \delta(r)$ in $\mathrm{Re}\, s \geq 0$, for $|r| < 1$.
Hence

$$\Phi(r,s)\frac{\lambda - s - r\lambda\beta(s)}{s - \delta(r)} = [\frac{r}{1-r} - r\Psi(r,s)]\frac{\lambda - s}{s - \delta(r)}.$$

LHS: analytic for $\mathrm{Re}\, s > 0$; bounded. RHS: analytic for $\mathrm{Re}\, s < 0$; bounded.

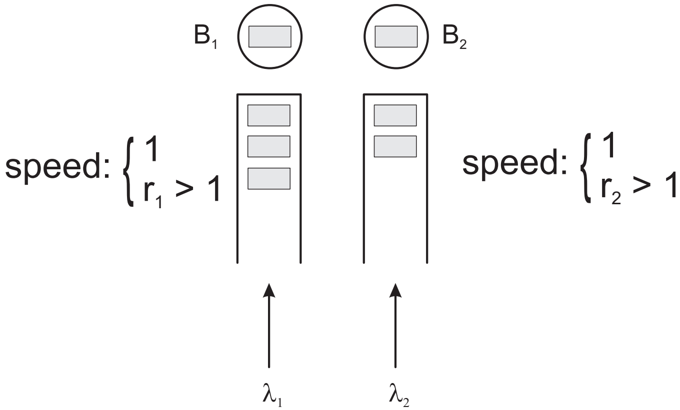
$$\Phi(r,s)\frac{\lambda - s - r\lambda\beta(s)}{s-\delta(r)} = [\frac{r}{1-r} - r\Psi(r,s)]\frac{\lambda - s}{s-\delta(r)}.$$

LHS: analytic for $\mathrm{Re}\ s > 0$; bounded. RHS: analytic for $\mathrm{Re}\ s < 0$: bounded.

Liouville's theorem: both sides equal some function $D(r)$.
Taking $s = 0$ shows that $D(r) = -\frac{\lambda r}{\delta(r)}$.

## 2. Two coupled processors: The Wiener-Hopf boundary value technique

Example of the boundary value method for workloads: **Coupled processor model.**




Two $M/G/1$ queues, arrival rates $\lambda_1$, $\lambda_2$, general service requirements $B_1$, $B_2$. Coupled via speeds.

$V_i$ steady-state workload at processor $i$, $i = 1, 2$.

$$\Psi(s_1, s_2) := \mathbb{E}[\mathrm{e}^{-s_1 V_1 - s_2 V_2}], \quad \Psi_0 := \mathbb{P}(V_1 = 0, V_2 = 0),$$

$$\Psi_1(s_2) := \mathbb{E}[\mathrm{e}^{-s_2 V_2} I(V_1 = 0)], \quad \Psi_2(s_1) := \mathbb{E}[\mathrm{e}^{-s_1 V_1} I(V_2 = 0)].$$

Fundamental equation:

$$\begin{aligned} &\{\lambda_1(1 - \mathbb{E}[\mathrm{e}^{-s_1 B_1}]) - s_1 + \lambda_2(1 - \mathbb{E}[\mathrm{e}^{-s_2 B_2}]) - s_2\}\Psi(s_1, s_2) = \\ - \ &[(r_1 - 1)s_1 + (r_2 - 1)s_2]\Psi_0 \\ + \ &[(r_1 - 1)s_1 - s_2]\Psi_2(s_1) + [(r_2 - 1)s_2 - s_1]\Psi_1(s_2). \end{aligned} \tag{1}$$

Look for pairs of zeroes $(s_1, s_2)$ with $\mathrm{Re}\, s_1, s_2 \geq 0$ of the *kernel* $K(s_1, s_2) := \lambda_1(1 - \mathbb{E}[\mathrm{e}^{-s_1 B_1}]) - s_1 + \lambda_2(1 - \mathbb{E}[\mathrm{e}^{-s_2 B_2}]) - s_2$

$$K(s_1, s_2) = \lambda_1(1 - \mathbb{E}[e^{-s_1 B_1}]) - s_1 + \lambda_2(1 - \mathbb{E}[e^{-s_2 B_2}]) - s_2$$

is a *separable* kernel (makes sense!):

$$K(s_1, s_2) = \gamma_1(s_1) + \gamma_2(s_2) = \gamma_1(s_1) + w + \gamma_2(s_2) - w.$$

Uniformization method:
Look for zeroes of $\gamma_1(s_1) + w$ and of $\gamma_2(s_2) - w$.
Under certain conditions one may show that $\gamma_1(s_1) + w = 0$, for $\mathrm{Re}\, w \geq 0$, $w \neq 0$, has exactly one zero $s_1 = \delta_1(w)$ in $\mathrm{Re}\, s_1 \geq 0$, with multiplicity one; and $\delta_1(w)$ is analytic in $\mathrm{Re}\, w > 0$, continuous in $\mathrm{Re}\, w \geq 0$.
Similarly for $\gamma_2(s_2) - w$, but now with the $w$ signs reversed: $s_2 = \delta_2(w)$.
Because of the analyticity of $\Psi(s_1, s_2)$ for these zero pairs $(s_1, s_2)$, the right hand side of (1) must be zero for those $(s_1, s_2)$.

Hence, for $\frac{1}{r_1} + \frac{1}{r_2} \neq 1$ (there is a simpler approach if $\frac{1}{r_1} + \frac{1}{r_2} = 1$), for $\mathrm{Re}\, w = 0$:

$$[(1 - \frac{1}{r_1})\delta_1(w) - \frac{1}{r_1}\delta_2(w)][\frac{1}{r_2}(\Psi_2(\delta_1(w)) - \Psi_0) - \frac{\Psi_0}{r_1 r_2}\frac{1}{1 - \frac{1}{r_1} - \frac{1}{r_2}}]$$
$$= -[(1 - \frac{1}{r_2})\delta_2(w) - \frac{1}{r_2}\delta_1(w)][\frac{1}{r_1}(\Psi_1(\delta_2(w)) - \Psi_0) - \frac{\Psi_0}{r_1 r_2}\frac{1}{1 - \frac{1}{r_1} - \frac{1}{r_2}}], \tag{2}$$

Observe that $\Psi_1(\delta_2(w))$ is analytic for $\mathrm{Re}\, w < 0$, continuous for $\mathrm{Re}\, w \leq 0$, whereas $\Psi_2(\delta_1(w))$ is analytic for $\mathrm{Re}\, w > 0$, continuous for $\mathrm{Re}\, w \geq 0$.

A careful analysis of the factors in front of the $\Psi$ functions in (2) reveals a way to factorize them as products of terms which are analytic in the left- and right half planes. One finally ends up with a Wiener-Hopf boundary value problem of the form: For $\mathrm{Re}\, w = 0$ (the *boundary*),

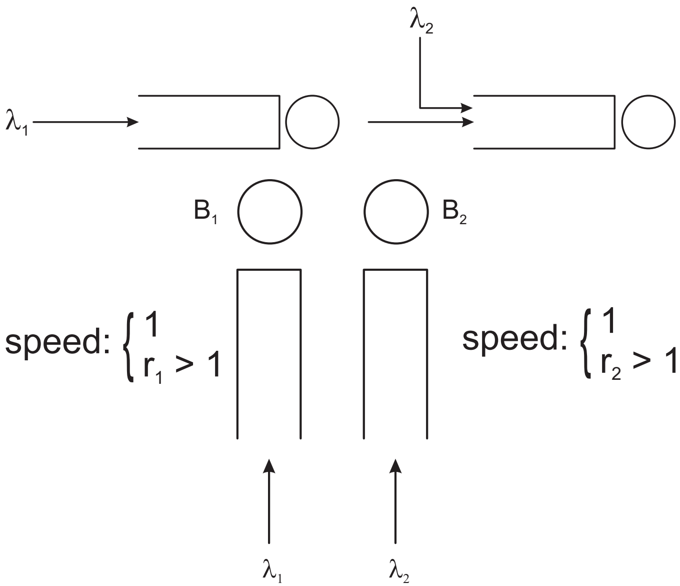
$$A_L(w)[\frac{1}{r_2}(\Psi_2(\delta_1(w)) - \Psi_0) - \frac{\Psi_0}{r_1 r_2}\frac{1}{1-\frac{1}{r_1}-\frac{1}{r_2}}] = -$$
$$A_R(w)[\frac{1}{r_1}(\Psi_1(\delta_2(w)) - \Psi_0) - \frac{\Psi_0}{r_1 r_2}\frac{1}{1-\frac{1}{r_1}-\frac{1}{r_2}}], \tag{3}$$

where the left hand side is analytic for $\mathrm{Re}\, w > 0$, continuous for $\mathrm{Re}\, w \geq 0$, and the right hand side is analytic for $\mathrm{Re}\, w < 0$, continuous for $\mathrm{Re}\, w \leq 0$.

So both sides are each other's analytic continuations.
Liouville's theorem: both sides are constants.
This determines $\Psi_1(\delta_2(w))$ for $\mathrm{Re}\, w \leq 0$ and $\Psi_2(\delta_1(w))$ for $\mathrm{Re}\, w \geq 0$, and next $\Psi_1(s_2)$ for $\mathrm{Re}\, s_2 \geq 0$ and $\Psi_2(s_1)$ for $\mathrm{Re}\, s_1 \geq 0$ (this may first require analytic continuation).

Finally, $\Psi(s_1, s_2)$ follows from (1).

The case $r_1 = 1,\ r_2 = 2$: relation to a tandem queue with *continuous* outflow from $Q_1$ and $Q_2$, and speeds $1$ and $2$.




*Same* two-dimensional workload process!

Generalization of the coupled processor model and of the tandem model: Lévy input process instead of compound Poisson (joint work with Jevgenijs Ivanovs)
(tandem queue: Miyazawa and Rolski (2009)).
The kernel

$$K(s_1, s_2) = \lambda_1(1 - \mathbb{E}[e^{-s_1 B_1}]) - s_1 + \lambda_2(1 - \mathbb{E}[e^{-s_2 B_2}]) - s_2$$

becomes:

$$K(s_1, s_2) = \gamma_1(s_1) + \gamma_2(s_2),$$

with $\gamma_i(s_i)$ the Laplace exponent of the $i$th Lévy process.
Again a separable kernel;
again look for zeros of $\gamma_2(s_2) - w = 0$ (and of $\gamma_1(s_1) + w = 0$).
This is the inverse of the Laplace exponent.
Relation to entrance time of the Lévy process:

$$\mathbb{E}[e^{-wT(u)}] = e^{-u\gamma_2^{-1}(w)} = e^{-u\delta_2(w)}.$$

## 4. Closing remarks: relation between the compensation approach and complex function techniques

Compensation approach (Ivo Adan) for a class of RW in first quadrant:
$p_{m,n} = \sum_{i=0}^{\infty} c_i \alpha_i^m \beta_i^n$.

This corresponds to
$P(x,y) = \sum_{i=0}^{\infty} \frac{c_i}{(1-\alpha_i x)(1-\beta_i y)}$,

hence $P(x,0) = \sum_{i=0}^{\infty} \frac{c_i}{1-\alpha_i x}$, $P(0,y) = \sum_{i=0}^{\infty} \frac{c_i}{1-\beta_i y}$.

These are meromorphic functions: the only singularities are isolated poles $\frac{1}{\alpha_i}, \frac{1}{\beta_i} \in C^-$.

The pairs $(\alpha_i, \beta_i)$ satisfy the balance equations in the interior.
The pairs $(\frac{1}{\alpha_i}, \frac{1}{\beta_i}) = (\hat{x}_i, \hat{y}_i)$ are zero pairs of the kernel $K(x, y)$.
Now consider the boundaries:

$$H(\hat{x}_i, \hat{y}_i)P(\hat{x}_i, 0) = -V(\hat{x}_i, \hat{y}_i)P(0, \hat{y}_i) - O(\hat{x}_i, \hat{y}_i)P(0, 0).$$

Each contribution of an $\hat{x}_i$ pole is compensated by a contribution of a $\hat{y}_i$ pole, which is subsequently compensated by a contribution of an $\hat{x}_{i+1}$ pole, etc.

For the tandem queue, the process stops after one term,
since $V(\hat{x}_0, \hat{y}_0) = 0$:

$$P(x, 0) = (1 - \rho_2)\frac{1 - \rho_1}{1 - \rho_1 x}.$$